Problem 11

In each of Problems 9 through 16, find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as t increases.

$$6y'' - 5y' + y = 0, \qquad y(0) = 4, \quad y'(0) = 0$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \to \quad y' = re^{rt} \quad \to \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$6(r^2 e^{rt}) - 5(re^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$6r^2 - 5r + 1 = 0$$

$$(3r - 1)(2r - 1) = 0$$

$$r = \left\{\frac{1}{3}, \frac{1}{2}\right\}$$

Two solutions to the ODE are $y = e^{t/3}$ and $y = e^{t/2}$, so the general solution is

$$y(t) = C_1 e^{t/3} + C_2 e^{t/2},$$

a linear combination of the two. Differentiate it once with respect to t.

$$y'(t) = \frac{C_1}{3} e^{t/3} + \frac{C_2}{2} e^{t/2}$$

Apply the two initial conditions now to determine C_1 and C_2.

$$y(0) = C_1 + C_2 = 4$$

$$y'(0) = \frac{C_1}{3} + \frac{C_2}{2} = 0$$

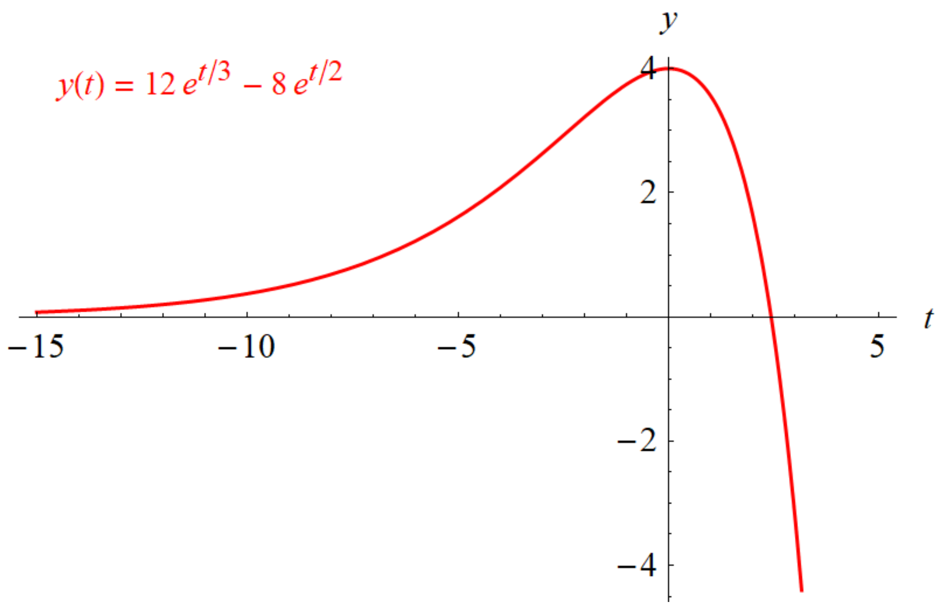
Solving the system of equations yields $C_1 = 12$ and $C_2 = -8$. Therefore,

$$y(t) = 12e^{t/3} - 8e^{t/2}.$$

Take the limit of $y(t)$ as $t \to \infty$.

$$\begin{aligned}
\lim_{t\to\infty} y(t) &= \lim_{t\to\infty} (12e^{t/3} - 8e^{t/2}) \\
&= \lim_{t\to\infty} 4e^{t/2}(3e^{-t/6} - 2) \\
&= \lim_{t\to\infty} 4e^{t/2}(-2) \\
&= \lim_{t\to\infty} -8e^{t/2}
\end{aligned}$$

This solution diverges to $-\infty$ as $t \to \infty$.


$y(t) = 12\,e^{t/3} - 8\,e^{t/2}$
y
t
4
2
−2
−4
−15
−10
−5
5