Problem 13

In each of Problems 9 through 16, find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as t increases.

$$y'' + 5y' + 3y = 0, \qquad y(0) = 1, \quad y'(0) = 0$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 5(re^{rt}) + 3(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 5r + 3 = 0$$

$$r = \frac{-5 \pm \sqrt{25 - 4(3)(1)}}{2} = \frac{-5 \pm \sqrt{13}}{2}$$

$$r = \left\{ \frac{-5 - \sqrt{13}}{2}, \frac{-5 + \sqrt{13}}{2} \right\}$$

Two solutions to the ODE are

$$y = \exp\left(\frac{-5 - \sqrt{13}}{2} t\right) \quad \text{and} \quad y = \exp\left(\frac{-5 + \sqrt{13}}{2} t\right),$$

so the general solution is

$$y(t) = C_1 \exp\left(\frac{-5 - \sqrt{13}}{2} t\right) + C_2 \exp\left(\frac{-5 + \sqrt{13}}{2} t\right),$$

a linear combination of the two. Differentiate it once with respect to t.

$$y'(t) = C_1 \left(\frac{-5 - \sqrt{13}}{2}\right) \exp\left(\frac{-5 - \sqrt{13}}{2} t\right) + C_2 \left(\frac{-5 + \sqrt{13}}{2}\right) \exp\left(\frac{-5 + \sqrt{13}}{2} t\right),$$

Apply the two initial conditions now to determine C_1 and C_2.

$$y(0) = C_1 + C_2 = 1$$

$$y'(0) = C_1 \left(\frac{-5 - \sqrt{13}}{2}\right) + C_2 \left(\frac{-5 + \sqrt{13}}{2}\right) = 0$$

Solving the system of equations yields

$$C_1 = \frac{1}{2} - \frac{5\sqrt{13}}{26}$$

$$C_2 = \frac{1}{2} + \frac{5\sqrt{13}}{26}.$$

Therefore,

$$y(t) = \left(\frac{1}{2} - \frac{5\sqrt{13}}{26}\right)\exp\left(\frac{-5-\sqrt{13}}{2}t\right) + \left(\frac{1}{2} + \frac{5\sqrt{13}}{26}\right)\exp\left(\frac{-5+\sqrt{13}}{2}t\right).$$

Because the coefficient of t in each exponential function is negative, this solution converges to 0 as $t \to \infty$.