Problem 16

In each of Problems 9 through 16, find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as t increases.

$$4y'' - y = 0, \qquad y(-2) = 1, \quad y'(-2) = -1$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$4(r^2 e^{rt}) - e^{rt} = 0$$

Divide both sides by e^{rt}.

$$4r^2 - 1 = 0$$
$$(2r+1)(2r-1) = 0$$
$$r = \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

Two solutions to the ODE are $y = e^{-t/2}$ and $y = e^{t/2}$, so the general solution is

$$y(t) = C_1 e^{-t/2} + C_2 e^{t/2},$$

a linear combination of the two. Differentiate it once with respect to t.

$$y'(t) = -\frac{C_1}{2} e^{-t/2} + \frac{C_2}{2} e^{t/2}$$

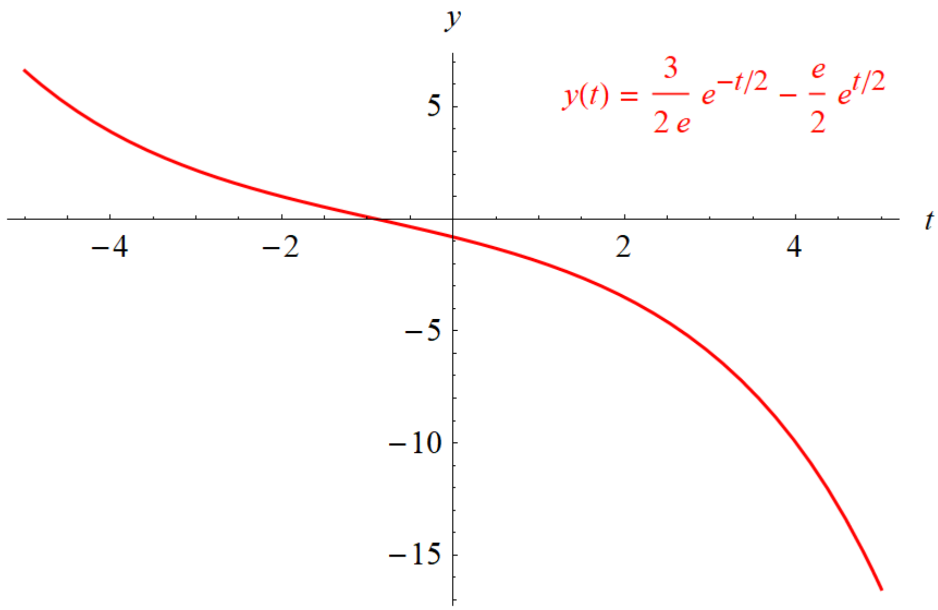
Apply the two initial conditions now to determine C_1 and C_2.

$$y(-2) = C_1 e^1 + C_2 e^{-1} = 1$$
$$y'(-2) = -\frac{C_1}{2} e^1 + \frac{C_2}{2} e^{-1} = -1$$

Solving the system of equations yields $C_1 = 3/(2e)$ and $C_2 = -e/2$. Therefore,

$$y(t) = \frac{3}{2e} e^{-t/2} - \frac{e}{2} e^{t/2}.$$

Because the coefficient of t in the second exponential function is positive, this solution diverges to $-\infty$ as $t \to \infty$.


y
$y(t) = \frac{3}{2e} e^{-t/2} - \frac{e}{2} e^{t/2}$
5
−4
−2
2
4
t
−5
−10
−15