Problem 21

Solve the initial value problem $y'' - y' - 2y = 0$, $y(0) = \alpha$, $y'(0) = 2$. Then find α so that the solution approaches zero as $t \to \infty$.

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \to \quad y' = re^{rt} \quad \to \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - re^{rt} - 2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1) = 0$$

$$r = \{-1, 2\}$$

Two solutions to the ODE are $y = e^{-t}$ and $y = e^{2t}$, so the general solution is

$$y(t) = C_1 e^{-t} + C_2 e^{2t},$$

a linear combination of the two. Differentiate it once with respect to t.

$$y'(t) = -C_1 e^{-t} + 2C_2 e^{2t}$$

Apply the two initial conditions now to determine C_1 and C_2.

$$y(0) = C_1 + C_2 = \alpha$$
$$y'(0) = -C_1 + 2C_2 = 2$$

Solving the system of equations yields

$$C_1 = \frac{2}{3}(\alpha - 1)$$
$$C_2 = \frac{1}{3}(\alpha + 2).$$

Therefore,

$$y(t) = \frac{2}{3}(\alpha - 1)e^{-t} + \frac{1}{3}(\alpha + 2)e^{2t}.$$

To prevent the solution from blowing up as $t \to \infty$, set $\alpha = -2$.