Problem 25

Consider the initial value problem

$$2y'' + 3y' - 2y = 0, \qquad y(0) = 1, \quad y'(0) = -\beta,$$

where $\beta > 0$.

(a) Solve the initial value problem.

(b) Plot the solution when $\beta = 1$. Find the coordinates (t_0, y_0) of the minimum point of the solution in this case.

(c) Find the smallest value of β for which the solution has no minimum point.

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$2(r^2 e^{rt}) + 3(re^{rt}) - 2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$2r^2 + 3r - 2 = 0$$
$$(2r - 1)(r + 2) = 0$$
$$r = \left\{-2, \frac{1}{2}\right\}$$

Two solutions to the ODE are $y = e^{-2t}$ and $y = e^{t/2}$. Therefore, the general solution is

$$y(t) = C_1 e^{-2t} + C_2 e^{t/2},$$

a linear combination of the two. Differentiate it once with respect to t.

$$y'(t) = -2C_1 e^{-2t} + \frac{C_2}{2} e^{t/2}$$

Apply the two initial conditions now to determine C_1 and C_2.

$$y(0) = C_1 + C_2 = 1$$
$$y'(0) = -2C_1 + \frac{C_2}{2} = -\beta$$

Solving this system of equations yields

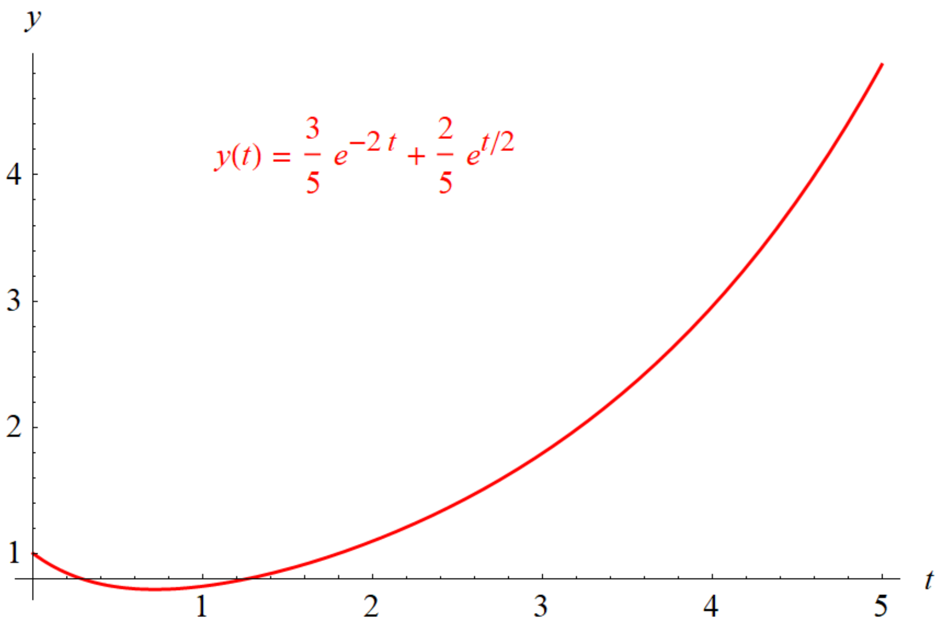
$$C_1 = \frac{1}{5}(1 + 2\beta)$$
$$C_2 = \frac{2}{5}(2 - \beta).$$

Therefore,

$$y(t) = \frac{1}{5}(1 + 2\beta)e^{-2t} + \frac{2}{5}(2 - \beta)e^{t/2}.$$

In the case that $\beta = 1$, the solution becomes

$$y(t) = \frac{3}{5}e^{-2t} + \frac{2}{5}e^{t/2}.$$



To find where the minimum is, solve $y'(t) = 0$ for t.

$$y'(t) = -\frac{6}{5}e^{-2t} + \frac{1}{5}e^{t/2} = 0$$

Divide both sides by $e^{t/2}$.

$$-\frac{6}{5}e^{-5t/2} + \frac{1}{5} = 0$$

$$-6e^{-5t/2} + 1 = 0$$

$$6e^{-5t/2} = 1$$

$$e^{-5t/2} = \frac{1}{6}$$

$$-\frac{5t}{2} = \ln\frac{1}{6}$$

$$\begin{aligned} t &= -\frac{2}{5}\ln\frac{1}{6} \\ &= \frac{2}{5}\ln 6 \approx 0.717 \end{aligned}$$

Therefore, the minimum is at

$$y\left(\frac{2}{5}\ln 6\right) = \frac{3^{1/5}}{5 \cdot 2^{4/5}} + \frac{2 \cdot 6^{1/5}}{5} \approx 0.715.$$

Return to the original solution.

$$y(t) = \frac{1}{5}(1 + 2\beta)e^{-2t} + \frac{2}{5}(2 - \beta)e^{t/2}.$$

Take a derivative with respect to t.

$$y'(t) = -\frac{2}{5}(1 + 2\beta)e^{-2t} + \frac{1}{5}(2 - \beta)e^{t/2}.$$

The graph of $y(t)$ has a minimum if $0 < \beta < 2$, so the smallest value of β for which there is no minimum is $\beta = 2$.