

Problem 20

If the Wronskian of f and g is $t \cos t - \sin t$, and if $u = f + 3g$, $v = f - g$, find the Wronskian of u and v .

Solution

Suppose $W(f, g)$ is the Wronskian of f and g . Then

$$W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - gf' = t \cos t - \sin t.$$

Now consider $W(u, v)$, the Wronskian of u and v .

$$\begin{aligned} W(u, v) &= \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} \\ &= \begin{vmatrix} f + 3g & f - g \\ f' + 3g' & f' - g' \end{vmatrix} \\ &= (f + 3g)(f' - g') - (f - g)(f' + 3g') \\ &= \cancel{ff'} - fg' + 3f'g - \cancel{3gg'} - \cancel{ff'} - 3fg' + f'g + \cancel{3gg'} \\ &= 4f'g - 4fg' \\ &= -4(fg' - f'g) \\ &= -4W(f, g) \end{aligned}$$

Therefore,

$$W(u, v) = -4(t \cos t - \sin t).$$