Problem 21

Assume that y_1 and y_2 are a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$ and let $y_3 = a_1y_1 + a_2y_2$ and $y_4 = b_1y_1 + b_2y_2$, where a_1, a_2, b_1, and b_2 are any constants. Show that

$$W(y_3, y_4) = (a_1b_2 - a_2b_1)W(y_1, y_2).$$

Are y_3 and y_4 also a fundamental set of solutions? Why or why not?

Solution

Suppose $W(y_1, y_2)$ is the Wronskian of y_1 and y_2. Then

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'.$$

Now consider $W(y_3, y_4)$, the Wronskian of y_3 and y_4.

$$\begin{aligned}
W(y_3, y_4) &= \begin{vmatrix} y_3 & y_4 \\ y_3' & y_4' \end{vmatrix} \\
&= \begin{vmatrix} a_1y_1 + a_2y_2 & b_1y_1 + b_2y_2 \\ a_1y_1' + a_2y_2' & b_1y_1' + b_2y_2' \end{vmatrix} \\
&= (a_1y_1 + a_2y_2)(b_1y_1' + b_2y_2') - (b_1y_1 + b_2y_2)(a_1y_1' + a_2y_2') \\
&= \cancel{a_1b_1y_1y_1'} + a_1b_2y_1y_2' + a_2b_1y_1'y_2 + \cancel{a_2b_2y_2y_2'} - \cancel{a_1b_1y_1y_1'} - a_2b_1y_1y_2' - a_1b_2y_1'y_2 - \cancel{a_2b_2y_2y_2'} \\
&= (a_1b_2 - a_2b_1)y_1y_2' + (a_2b_1 - a_1b_2)y_1'y_2 \\
&= (a_1b_2 - a_2b_1)y_1y_2' - (a_1b_2 - a_2b_1)y_1'y_2 \\
&= (a_1b_2 - a_2b_1)(y_1y_2' - y_1'y_2) \\
&= (a_1b_2 - a_2b_1)W(y_1, y_2)
\end{aligned}$$

Because y_1 and y_2 form a fundamental set of solutions, $W(y_1, y_2) \neq 0$. Provided that the Wronskian of y_3 and y_4 is nonzero, that is,

$$a_1b_2 - a_2b_1 \neq 0,$$

y_3 and y_4 form a fundamental set of solutions as well.