Problem 22

In each of Problems 22 and 23, find the fundamental set of solutions specified by Theorem 3.2.5 for the given differential equation and initial point.

$$y'' + y' - 2y = 0, \qquad t_0 = 0$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + re^{rt} - 2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + r - 2 = 0$$
$$(r+2)(r-1) = 0$$
$$r = \{-2, 1\}$$

Two solutions to the ODE are $y = e^{-2t}$ and $y = e^t$, so the general solution is

$$y(t) = C_1 e^{-2t} + C_2 e^t,$$

a linear combination of the two. According to Theorem 3.2.5, y_1 satisfies the initial conditions,

$$y(t_0) = 1, \qquad y'(t_0) = 0,$$

and y_2 satisfies the initial conditions,

$$y(t_0) = 0, \qquad y'(t_0) = 1.$$

Differentiate the general solution once with respect to t.

$$y'(t) = -2C_1 e^{-2t} + C_2 e^t$$

The two systems of equations are then

$$\begin{aligned} y_1(0) &= C_1 + C_2 = 1 \\ y_1'(0) &= -2C_1 + C_2 = 0 \end{aligned} \qquad\qquad \begin{aligned} y_2(0) &= C_1 + C_2 = 0 \\ y_2'(0) &= -2C_1 + C_2 = 1. \end{aligned}$$

Solving the one for y_1 yields $C_1 = 1/3$ and $C_2 = 2/3$, and solving the one for y_2 yields $C_1 = -1/3$ and $C_2 = 1/3$. Therefore,

$$\begin{aligned} y_1(t) &= \frac{1}{3}e^{-2t} + \frac{2}{3}e^t \\ y_2(t) &= -\frac{1}{3}e^{-2t} + \frac{1}{3}e^t. \end{aligned}$$