Problem 25

In each of Problems 24 through 27, verify that the functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions?

$$y'' - 2y' + y = 0; \qquad y_1(t) = e^t, \quad y_2(t) = te^t$$

Solution

Check that y_1 is a solution of the ODE.

$$y_1'' - 2y_1' + y_1 \stackrel{?}{=} 0$$

$$\frac{d^2}{dt^2}(e^t) - 2\frac{d}{dt}(e^t) + e^t \stackrel{?}{=} 0$$

$$e^t - 2e^t + e^t \stackrel{?}{=} 0$$

$$0 = 0$$

Now check that y_2 is a solution of the ODE.

$$y_2'' - 2y_2' + y_2 \stackrel{?}{=} 0$$

$$\frac{d^2}{dt^2}(te^t) - 2\frac{d}{dt}(te^t) + te^t \stackrel{?}{=} 0$$

$$\frac{d}{dt}(e^t + te^t) - 2(e^t + te^t) + te^t \stackrel{?}{=} 0$$

$$e^t + e^t + te^t - 2e^t - 2te^t + te^t \stackrel{?}{=} 0$$

$$0 = 0$$

Calculate $W(y_1, y_2)$, the Wronskian of y_1 and y_2.

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} \\ &= e^t(e^t + te^t) - te^t(e^t) \\ &= e^{2t} + te^{2t} - te^{2t} \\ &= e^{2t} \end{aligned}$$

Since $W(y_1, y_2) \neq 0$, y_1 and y_2 form a fundamental set of solutions.