Problem 29

In each of Problems 29 through 32, find the Wronskian of two solutions of the given differential equation without solving the equation.

$$t^2y'' - t(t+2)y' + (t+2)y = 0$$

Solution

Suppose that y_1 and y_2 are two solutions to the ODE. They then satisfy

$$\begin{aligned} t^2y_1'' - t(t+2)y_1' + (t+2)y_1 &= 0 \\ t^2y_2'' - t(t+2)y_2' + (t+2)y_2 &= 0. \end{aligned}$$

Multiply both sides of the first equation by $-y_2$ and multiply both sides of the second equation by y_1.

$$\begin{aligned} -t^2y_1''y_2 + t(t+2)y_1'y_2 - (t+2)y_1y_2 &= 0 \\ t^2y_1y_2'' - t(t+2)y_1y_2' + (t+2)y_1y_2 &= 0 \end{aligned}$$

Add the respective sides of each equation.

$$-t^2y_1''y_2 + t^2y_1y_2'' + t(t+2)y_1'y_2 - t(t+2)y_1y_2' = 0$$

Factor the left side.

$$t^2(y_1y_2'' - y_1''y_2) + t(t+2)(y_1'y_2 - y_1y_2') = 0$$

Note that the Wronskian of y_1 and y_2 is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 \quad \Rightarrow \quad W'(y_1, y_2) = \cancel{y_1'y_2'} + y_1y_2'' - y_1''y_2 - \cancel{y_1'y_2'} = y_1y_2'' - y_1''y_2,$$

so the previous equation can be written as

$$t^2W' - t(t+2)W = 0.$$

Solve for W by separating variables.

$$\frac{dW}{W} = \frac{t+2}{t}\,dt$$

Integrate both sides.

$$\ln|W| = t + 2\ln t + C$$

Exponentiate both sides.

$$\begin{aligned} |W| &= e^{t+2\ln t + C} \\ &= e^t e^{\ln t^2} e^C \\ &= e^C t^2 e^t \end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$W(t) = \pm e^C t^2 e^t$$

Therefore, using a new constant A for $\pm e^C$, the Wronskian of two solutions of the ODE is

$$W(t) = At^2e^t.$$