Problem 30

In each of Problems 29 through 32, find the Wronskian of two solutions of the given differential equation without solving the equation.

$$(\cos t)y'' + (\sin t)y' - ty = 0$$

Solution

Suppose that y_1 and y_2 are two solutions to the ODE. They then satisfy

$$\begin{aligned}(\cos t)y_1'' + (\sin t)y_1' - ty_1 = 0\\ (\cos t)y_2'' + (\sin t)y_2' - ty_2 = 0.\end{aligned}$$

Multiply both sides of the first equation by $-y_2$ and multiply both sides of the second equation by y_1.

$$\begin{aligned}-(\cos t)y_1''y_2 - (\sin t)y_1'y_2 + ty_1y_2 = 0\\ (\cos t)y_1y_2'' + (\sin t)y_1y_2' - ty_1y_2 = 0\end{aligned}$$

Add the respective sides of each equation.

$$-(\cos t)y_1''y_2 + (\cos t)y_1y_2'' - (\sin t)y_1'y_2 + (\sin t)y_1y_2' = 0$$

Factor the left side.

$$(\cos t)(y_1y_2'' - y_1''y_2) + (\sin t)(y_1y_2' - y_1'y_2) = 0$$

Note that the Wronskian of y_1 and y_2 is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 \quad \Rightarrow \quad W'(y_1, y_2) = \cancel{y_1'y_2'} + y_1y_2'' - y_1''y_2 - \cancel{y_1'y_2'} = y_1y_2'' - y_1''y_2,$$

so the previous equation can be written as

$$(\cos t)W' + (\sin t)W = 0$$

Solve for W by separating variables.

$$\frac{dW}{W} = -\frac{\sin t}{\cos t}\,dt$$

Integrate both sides.

$$\begin{aligned}\ln|W| &= \ln|\cos t| + C\\ \ln|W| - \ln|\cos t| &= C\\ \ln\left|\frac{W}{\cos t}\right| &= C\end{aligned}$$

Exponentiate both sides.

$$\left|\frac{W}{\cos t}\right| = e^C$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$\frac{W}{\cos t} = \pm e^C$$

Therefore, using a new constant A for $\pm e^C$, the Wronskian of two solutions of the ODE is

$$W(t) = A\cos t.$$