Problem 31

In each of Problems 29 through 32, find the Wronskian of two solutions of the given differential equation without solving the equation.

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \qquad \text{Bessel's equation}$$

Solution

Suppose that y_1 and y_2 are two solutions to the ODE. They then satisfy

$$\begin{aligned} x^2y_1'' + xy_1' + (x^2 - \nu^2)y_1 &= 0 \\ x^2y_2'' + xy_2' + (x^2 - \nu^2)y_2 &= 0. \end{aligned}$$

Multiply both sides of the first equation by $-y_2$ and multiply both sides of the second equation by y_1.

$$\begin{aligned} -x^2y_1''y_2 - xy_1'y_2 - (x^2 - \nu^2)y_1y_2 &= 0 \\ x^2y_1y_2'' + xy_1y_2' + (x^2 - \nu^2)y_1y_2 &= 0 \end{aligned}$$

Add the respective sides of each equation.

$$-x^2y_1''y_2 + x^2y_1y_2'' - xy_1'y_2 + xy_1y_2' = 0$$

Factor the left side.

$$x^2(y_1y_2'' - y_1''y_2) + x(y_1y_2' - y_1'y_2) = 0$$

Note that the Wronskian of y_1 and y_2 is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 \quad \Rightarrow \quad W'(y_1, y_2) = \cancel{y_1'y_2'} + y_1y_2'' - y_1''y_2 - \cancel{y_1'y_2'} = y_1y_2'' - y_1''y_2,$$

so the previous equation can be written as

$$x^2W' + xW = 0.$$

Solve for W by separating variables.

$$\frac{dW}{W} = -\frac{dx}{x}$$

Integrate both sides.

$$\begin{aligned} \ln|W| &= -\ln|x| + C \\ \ln|W| + \ln|x| &= C \\ \ln|xW| &= C \end{aligned}$$

Exponentiate both sides.

$$|xW| = e^C$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$xW = \pm e^C$$

Therefore, using a new constant A for $\pm e^C$, the Wronskian of two solutions of the ODE is

$$W(x) = \frac{A}{x}.$$