Problem 32

In each of Problems 29 through 32, find the Wronskian of two solutions of the given differential equation without solving the equation.

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \qquad \text{Legendre's equation}$$

Solution

Suppose that y_1 and y_2 are two solutions to the ODE. They then satisfy

$$\begin{aligned}(1-x^2)y_1'' - 2xy_1' + \alpha(\alpha+1)y_1 &= 0\\ (1-x^2)y_2'' - 2xy_2' + \alpha(\alpha+1)y_2 &= 0.\end{aligned}$$

Multiply both sides of the first equation by $-y_2$ and multiply both sides of the second equation by y_1.

$$\begin{aligned}-(1-x^2)y_1''y_2 + 2xy_1'y_2 - \alpha(\alpha+1)y_1y_2 &= 0\\ (1-x^2)y_1y_2'' - 2xy_1y_2' + \alpha(\alpha+1)y_1y_2 &= 0\end{aligned}$$

Add the respective sides of each equation.

$$-(1-x^2)y_1''y_2 + (1-x^2)y_1y_2'' + 2xy_1'y_2 - 2xy_1y_2' = 0$$

Factor the left side.

$$(1-x^2)(y_1y_2'' - y_1''y_2) - 2x(y_1y_2' - y_1'y_2) = 0$$

Note that the Wronskian of y_1 and y_2 is

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 \quad \Rightarrow \quad W'(y_1,y_2) = \cancel{y_1'y_2'} + y_1y_2'' - y_1''y_2 - \cancel{y_1'y_2'} = y_1y_2'' - y_1''y_2,$$

so the previous equation can be written as

$$(1-x^2)W' - 2xW = 0.$$

Solve for W by separating variables.

$$\frac{dW}{W} = \frac{2x}{1-x^2}\,dx$$

Integrate both sides.

$$\begin{aligned}\ln|W| &= -\ln|1-x^2| + C\\ \ln|W| + \ln|1-x^2| &= C\\ \ln|(1-x^2)W| &= C\end{aligned}$$

Exponentiate both sides.

$$|(1-x^2)W| = e^C$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$(1-x^2)W = \pm e^C$$

Therefore, using a new constant A for $\pm e^C$, the Wronskian of two solutions of the ODE is

$$W(x) = \frac{A}{1-x^2}.$$