Problem 33

Show that if p is differentiable and $p(t) > 0$, then the Wronskian $W(t)$ of two solutions of $[p(t)y']' + q(t)y = 0$ is $W(t) = c/p(t)$, where c is a constant.

Solution

Suppose that y_1 and y_2 are two solutions to the ODE. They then satisfy

$$\begin{aligned} [p(t)y_1']' + q(t)y_1 &= 0 \\ [p(t)y_2']' + q(t)y_2 &= 0. \end{aligned}$$

Multiply both sides of the first equation by $-y_2$ and multiply both sides of the second equation by y_1.

$$\begin{aligned} -[p(t)y_1']'y_2 - q(t)y_1y_2 &= 0 \\ y_1[p(t)y_2']' + q(t)y_1y_2 &= 0 \end{aligned}$$

Add the respective sides of each equation.

$$-[p(t)y_1']'y_2 + y_1[p(t)y_2']' = 0$$

Expand the left side.

$$-p'(t)y_1'y_2 - p(t)y_1''y_2 + p'(t)y_1y_2' + p(t)y_1y_2'' = 0$$

Factor the left side.

$$p(t)(y_1y_2'' - y_1''y_2) + p'(t)(y_1y_2' - y_1'y_2) = 0$$

Note that the Wronskian of y_1 and y_2 is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 \quad \Rightarrow \quad W'(y_1, y_2) = \cancel{y_1'y_2'} + y_1y_2'' - y_1''y_2 - \cancel{y_1'y_2'} = y_1y_2'' - y_1''y_2,$$

so the previous equation can be written as

$$p(t)W' + p'(t)W = 0.$$

The left side is the derivative of the product $p(t)W$.

$$\frac{d}{dt}[p(t)W] = 0$$

Integrate both sides with respect to t.

$$p(t)W = C$$

Therefore, the Wronskian $W(t)$ of two solutions of the ODE is

$$W(t) = \frac{C}{p(t)}.$$