Problem 35

If the differential equation $t^2y'' - 2y' + (3+t)y = 0$ has y_1 and y_2 as a fundamental set of solutions and if $W(y_1, y_2)(2) = 3$, find the value of $W(y_1, y_2)(4)$.

Solution

Since y_1 and y_2 are both solutions to the ODE, they satisfy

$$\begin{aligned} t^2y_1'' - 2y_1' + (3+t)y_1 &= 0 \\ t^2y_2'' - 2y_2' + (3+t)y_2 &= 0. \end{aligned}$$

Multiply both sides of the first equation by $-y_2$ and multiply both sides of the second equation by y_1.

$$\begin{aligned} -t^2y_1''y_2 + 2y_1'y_2 - (3+t)y_1y_2 &= 0 \\ t^2y_1y_2'' - 2y_1y_2' + (3+t)y_1y_2 &= 0. \end{aligned}$$

Add the respective sides of each equation.

$$-t^2y_1''y_2 + t^2y_1y_2'' + 2y_1'y_2 - 2y_1y_2' = 0$$

Factor the left side.

$$t^2(y_1y_2'' - y_1''y_2) - 2(y_1y_2' - y_1'y_2) = 0$$

Note that the Wronskian of y_1 and y_2 is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 \quad \Rightarrow \quad W'(y_1, y_2) = \cancel{y_1'y_2'} + y_1y_2'' - y_1''y_2 - \cancel{y_1'y_2'} = y_1y_2'' - y_1''y_2,$$

so the previous equation can be written as

$$t^2W' - 2W = 0$$

$$t^2\frac{dW}{dt} = 2W.$$

Solve this ODE by separating variables.

$$\frac{dW}{W} = \frac{2}{t^2}\,dt$$

Integrate both sides.

$$\ln|W| = -\frac{2}{t} + C$$

Exponentiate both sides.

$$\begin{aligned} |W| &= \exp\left(-\frac{2}{t} + C\right) \\ &= e^Ce^{-2/t} \end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$W(t) = \pm e^Ce^{-2/t}$$

Using a new constant A for $\pm e^C$, the Wronskian of two solutions of the ODE is

$$W(t) = Ae^{-2/t}.$$

Use the fact that $W = 3$ when $t = 2$ to determine A.

$$W(2) = Ae^{-1} = 3 \quad \rightarrow \quad A = 3e$$

The Wronskian is then

$$\begin{aligned} W(t) &= (3e)e^{-2/t} \\ &= 3e^{1-2/t}. \end{aligned}$$

Therefore,

$$W(4) = 3e^{1/2} \approx 4.95.$$