Problem 36

If the Wronskian of any two solutions of $y'' + p(t)y' + q(t)y = 0$ is constant, what does this imply about the coefficients p and q?

Solution

Suppose that y_1 and y_2 are two solutions to the ODE. They then satisfy

$$\begin{aligned} y_1'' + p(t)y_1' + q(t)y_1 &= 0 \\ y_2'' + p(t)y_2' + q(t)y_2 &= 0. \end{aligned}$$

Multiply both sides of the first equation by $-y_2$ and multiply both sides of the second equation by y_1.

$$\begin{aligned} -y_1''y_2 - p(t)y_1'y_2 - q(t)y_1y_2 &= 0 \\ y_1y_2'' + p(t)y_1y_2' + q(t)y_1y_2 &= 0. \end{aligned}$$

Add the respective sides of each equation.

$$-y_1''y_2 + y_1y_2'' - p(t)y_1'y_2 + p(t)y_1y_2' = 0$$

Factor the left side.

$$y_1y_2'' - y_1''y_2 + p(t)(y_1y_2' - y_1'y_2) = 0$$

Note that the Wronskian of y_1 and y_2 is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 \quad \Rightarrow \quad W'(y_1, y_2) = \cancel{y_1'y_2'} + y_1y_2'' - y_1''y_2 - \cancel{y_1'y_2'} = y_1y_2'' - y_1''y_2,$$

so the previous equation can be written as

$$W' + p(t)W = 0.$$

If W is constant, then W' is zero.

$$p(t)W = 0$$

Therefore, assuming that W is nonzero and dividing both sides by W, $p(t) = 0$.