Problem 42

In each of Problems 42 through 45, use the result of Problem 41 to determine whether the given equation is exact. If it is, then solve the equation.

$$y'' + xy' + y = 0$$

Solution

Notice that the last two terms can be written as $(xy)'$.

$$y'' + (xy)' = 0$$

Integrate both sides with respect to x.

$$y' + xy = C_1$$

To solve this first-order linear inhomogeneous ODE, multiply both sides by an integrating factor I.

$$I = \exp\left(\int^x s\,ds\right) = e^{x^2/2}$$

Proceed with the multiplication.

$$e^{x^2/2}y' + xe^{x^2/2}y = C_1e^{x^2/2}$$

The left side can be written as $(e^{x^2/2}y)'$ by the product rule.

$$(e^{x^2/2}y)' = C_1e^{x^2/2}$$

Integrate both sides with respect to x once more.

$$e^{x^2/2}y = \int^x C_1e^{s^2/2}\,ds + C_2$$

Therefore,

$$\begin{aligned} y(x) &= e^{-x^2/2}\int^x C_1e^{s^2/2}\,ds + C_2e^{-x^2/2} \\ &= C_1\int^x e^{(s^2-x^2)/2}\,ds + C_2e^{-x^2/2}. \end{aligned}$$