Problem 44

In each of Problems 42 through 45, use the result of Problem 41 to determine whether the given equation is exact. If it is, then solve the equation.

$$xy'' - (\cos x)y' + (\sin x)y = 0, \quad x > 0$$

Solution

This ODE is exact if it can be written in the form,

$$(xy')' + [f(x)y]' = 0.$$

Expand both sides.

$$y' + xy'' + f'(x)y + f(x)y' = 0$$
$$xy'' + [1 + f(x)]y' + f'(x)y = 0$$

Comparing this with the original ODE, we see that

$$1 + f(x) = -\cos x$$
$$f'(x) = \sin x$$

Integrate both sides of the second equation with respect to x, and choose the constant of integration to be -1 to satisfy the first equation.

$$f(x) = -\cos x - 1$$

As a result, the original ODE is exact because it can be written as

$$(xy')' + [-(\cos x + 1)y]' = 0.$$

Integrate both sides with respect to x.

$$xy' + [-(\cos x + 1)y] = C_1$$

Divide both sides by x.

$$y' - \left(\frac{\cos x}{x} + \frac{1}{x}\right) y = \frac{C_1}{x}$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left[\int^x -\left(\frac{\cos s}{s} + \frac{1}{s}\right) ds\right]$$

Proceed with the multiplication.

$$\exp\left[-\int^x \left(\frac{\cos s}{s} + \frac{1}{s}\right) ds\right] y' - \left(\frac{\cos x}{x} + \frac{1}{x}\right) \exp\left[-\int^x \left(\frac{\cos s}{s} + \frac{1}{s}\right) ds\right] y = \frac{C_1}{x} \exp\left[-\int^x \left(\frac{\cos s}{s} + \frac{1}{s}\right) ds\right]$$

Use the product rule to write the left side as $d/dx(Iy)$.

$$\frac{d}{dx}\left\{\exp\left[-\int^x\left(\frac{\cos s}{s}+\frac{1}{s}\right)ds\right]y\right\}=\frac{C_1}{x}\exp\left[-\int^x\left(\frac{\cos s}{s}+\frac{1}{s}\right)ds\right]$$

Integrate both sides with respect to x once more.

$$\exp\left[-\int^x\left(\frac{\cos s}{s}+\frac{1}{s}\right)ds\right]y=\int^x\frac{C_1}{r}\exp\left[-\int^r\left(\frac{\cos s}{s}+\frac{1}{s}\right)ds\right]dr+C_2$$

Therefore,

$$\begin{aligned} y(x)=\exp\left[\int^x\left(\frac{\cos s}{s}+\frac{1}{s}\right)ds\right]\int^x\frac{C_1}{r}\exp\left[-\int^r\left(\frac{\cos s}{s}+\frac{1}{s}\right)ds\right]dr \\ +C_2\exp\left[\int^x\left(\frac{\cos s}{s}+\frac{1}{s}\right)ds\right]. \end{aligned}$$