Problem 15

Show that if $y = \phi(t)$ is a solution of the differential equation $y'' + p(t)y' + q(t)y = g(t)$, where $g(t)$ is not always zero, then $y = c\phi(t)$, where c is any constant other than 1, is not a solution. Explain why this result does not contradict the remark following Theorem 3.2.2.

Solution

Suppose that $\phi(t)$ is a solution of the ODE. Then

$$\phi'' + p(t)\phi' + q(t)\phi = g(t).$$

Multiply both sides by c.

$$c\phi'' + cp(t)\phi' + cq(t)\phi = cg(t)$$

$$(c\phi)'' + p(t)(c\phi)' + q(t)(c\phi) = cg(t)$$

Therefore, $y = c\phi(t)$ satisfies $y'' + p(t)y' + q(t)y = cg(t)$, not $y'' + p(t)y' + q(t)y = g(t)$.

This result does not contradict Theorem 3.2.2 because the theorem assumes the ODE is of the form $y'' + p(t)y' + q(t)y = 0$. The ODE dealt with in this problem is $y'' + p(t)y' + q(t)y = g(t)$, a different one, so Theorem 3.2.2 does not apply.