Problem 23

In each of Problems 22 and 23, find the fundamental set of solutions specified by Theorem 3.2.5 for the given differential equation and initial point.

$$y'' + 4y' + 3y = 0, \qquad t_0 = 1$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 4(re^{rt}) + 3(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 4r + 3 = 0$$

$$(r+3)(r+1) = 0$$

$$r = \{-3, -1\}$$

Two solutions to the ODE are $y = e^{-3t}$ and $y = e^{-t}$, so the general solution is

$$y(t) = C_1 e^{-3t} + C_2 e^{-t},$$

a linear combination of the two. According to Theorem 3.2.5, y_1 satisfies the initial conditions,

$$y(t_0) = 1, \qquad y'(t_0) = 0,$$

and y_2 satisfies the initial conditions,

$$y(t_0) = 0, \qquad y'(t_0) = 1.$$

Differentiate the general solution once with respect to t.

$$y'(t) = -3C_1 e^{-3t} - C_2 e^{-t}$$

The two systems of equations are then

$$\begin{aligned} y_1(1) &= C_1 e^{-3} + C_2 e^{-1} = 1 \\ y_1'(1) &= -3C_1 e^{-3} - C_2 e^{-1} = 0 \end{aligned} \qquad \begin{aligned} y_2(1) &= C_1 e^{-3} + C_2 e^{-1} = 0 \\ y_2'(1) &= -3C_1 e^{-3} - C_2 e^{-1} = 1. \end{aligned}$$

Solving the one for y_1 yields $C_1 = -e^3/2$ and $C_2 = 3e/2$, and solving the one for y_2 yields $C_1 = -e^3/2$ and $C_2 = e/2$. Therefore,

$$y_1(t) = -\frac{e^3}{2} e^{-3t} + \frac{3e}{2} e^{-t}$$

$$y_2(t) = -\frac{e^3}{2} e^{-3t} + \frac{e}{2} e^{-t}.$$