Problem 28

Consider the equation $y'' - y' - 2y = 0$.

(a) Show that $y_1(t) = e^{-t}$ and $y_2(t) = e^{2t}$ form a fundamental set of solutions.

(b) Let $y_3(t) = -2e^{2t}$, $y_4(t) = y_1(t) + 2y_2(t)$, and $y_5(t) = 2y_1(t) - 2y_3(t)$. Are $y_3(t)$, $y_4(t)$, and $y_5(t)$ also solutions of the given differential equation?

(c) Determine whether each of the following pairs forms a fundamental set of solutions: $[y_1(t), y_3(t)]$; $[y_2(t), y_3(t)]$; $[y_1(t), y_4(t)]$; $[y_4(t), y_5(t)]$.

Solution

Part (a)

Calculate $W(y_1, y_2)$, the Wronskian of y_1 and y_2.

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix} \\ &= e^{-t}(2e^{2t}) - e^{2t}(-e^{-t}) \\ &= 2e^t + e^t \\ &= 3e^t \end{aligned}$$

Since $W(y_1, y_2) \neq 0$, y_1 and y_2 form a fundamental set of solutions.

Part (b)

Check that y_3 is a solution of the ODE.

$$y_3'' - y_3' - 2y_3 \stackrel{?}{=} 0$$

$$\frac{d^2}{dt^2}(-2e^{2t}) - \frac{d}{dt}(-2e^{2t}) - 2(-2e^{2t}) \stackrel{?}{=} 0$$

$$(-8e^{2t}) - (-4e^{2t}) - 2(-2e^{2t}) \stackrel{?}{=} 0$$

$$-8e^{2t} + 4e^{2t} + 4e^{2t} \stackrel{?}{=} 0$$

$$0 = 0$$

Now check that $y_4 = e^{-t} + 2e^{2t}$ is a solution of the ODE.

$$y_4'' - y_4' - 2y_4 \stackrel{?}{=} 0$$

$$\frac{d^2}{dt^2}(e^{-t} + 2e^{2t}) - \frac{d}{dt}(e^{-t} + 2e^{2t}) - 2(e^{-t} + 2e^{2t}) \stackrel{?}{=} 0$$

$$(e^{-t} + 8e^{2t}) - (-e^{-t} + 4e^{2t}) - 2(e^{-t} + 2e^{2t}) \stackrel{?}{=} 0$$

$$\cancel{e^{-t}} + \cancel{8e^{2t}} + \cancel{e^{-t}} - \cancel{4e^{2t}} - \cancel{2e^{-t}} - \cancel{4e^{2t}} \stackrel{?}{=} 0$$

$$0 = 0$$

Now check that $y_5 = 2y_1(t) - 2y_3(t) = 2e^{-t} - 2(-2e^{2t}) = 2e^{-t} + 4e^{2t}$ is a solution of the ODE.

$$y_5'' - y_5' - 2y_5 \stackrel{?}{=} 0$$

$$\frac{d^2}{dt^2}(2e^{-t} + 4e^{2t}) - \frac{d}{dt}(2e^{-t} + 4e^{2t}) - 2(2e^{-t} + 4e^{2t}) \stackrel{?}{=} 0$$

$$(2e^{-t} + 16e^{2t}) - (-2e^{-t} + 8e^{2t}) - 2(2e^{-t} + 4e^{2t}) \stackrel{?}{=} 0$$

$$\cancel{2e^{-t}} + \cancel{16e^{2t}} + \cancel{2e^{-t}} - \cancel{8e^{2t}} - \cancel{4e^{-t}} - \cancel{8e^{2t}} \stackrel{?}{=} 0$$

$$0 = 0$$

Part (c)

Calculate $W(y_1, y_3)$, the Wronskian of y_1 and y_3.

$$\begin{aligned} W(y_1, y_3) &= \begin{vmatrix} y_1 & y_3 \\ y_1' & y_3' \end{vmatrix} \\ &= \begin{vmatrix} e^{-t} & -2e^{2t} \\ -e^{-t} & -4e^{2t} \end{vmatrix} \\ &= e^{-t}(-4e^{2t}) - (-2e^{2t})(-e^{-t}) \\ &= -4e^{t} - 2e^{t} \\ &= -6e^{t} \end{aligned}$$

Since $W(y_1, y_3) \neq 0$, y_1 and y_3 form a fundamental set of solutions.

Now calculate $W(y_2, y_3)$, the Wronskian of y_2 and y_3.

$$\begin{aligned}
W(y_2, y_3) &= \begin{vmatrix} y_2 & y_3 \\ y_2' & y_3' \end{vmatrix} \\
&= \begin{vmatrix} e^{2t} & -2e^{2t} \\ 2e^{2t} & -4e^{2t} \end{vmatrix} \\
&= e^{2t}(-4e^{2t}) - (-2e^{2t})(2e^{2t}) \\
&= -4e^{4t} + 4e^{4t} \\
&= 0
\end{aligned}$$

Since $W(y_2, y_3) = 0$, y_2 and y_3 do not form a fundamental set of solutions. Now calculate $W(y_1, y_4)$, the Wronskian of y_1 and y_4.

$$\begin{aligned}
W(y_1, y_4) &= \begin{vmatrix} y_1 & y_4 \\ y_1' & y_4' \end{vmatrix} \\
&= \begin{vmatrix} e^{-t} & e^{-t} + 2e^{2t} \\ -e^{-t} & -e^{-t} + 4e^{2t} \end{vmatrix} \\
&= e^{-t}(-e^{-t} + 4e^{2t}) - (e^{-t} + 2e^{2t})(-e^{-t}) \\
&= -e^{-2t} + 4e^{t} + e^{-2t} + 2e^{t} \\
&= 6e^{t}
\end{aligned}$$

Since $W(y_1, y_4) \neq 0$, y_1 and y_4 form a fundamental set of solutions. Now calculate $W(y_4, y_5)$, the Wronskian of y_4 and y_5.

$$\begin{aligned}
W(y_4, y_5) &= \begin{vmatrix} y_4 & y_5 \\ y_4' & y_5' \end{vmatrix} \\
&= \begin{vmatrix} e^{-t} + 2e^{2t} & 2e^{-t} + 4e^{2t} \\ -e^{-t} + 4e^{2t} & -2e^{-t} + 8e^{2t} \end{vmatrix} \\
&= (e^{-t} + 2e^{2t})(-2e^{-t} + 8e^{2t}) - (2e^{-t} + 4e^{2t})(-e^{-t} + 4e^{2t}) \\
&= -2e^{-2t} + 8e^{t} - 4e^{t} + 16e^{4t} - (-2e^{-2t} + 8e^{t} - 4e^{t} + 16e^{4t}) \\
&= 0
\end{aligned}$$

Since $W(y_4, y_5) = 0$, y_4 and y_5 do not form a fundamental set of solutions.