Problem 41

Exact Equations. The equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

is said to be exact if it can be written in the form

$$[P(x)y']' + [f(x)y]' = 0,$$

where $f(x)$ is to be determined in terms of $P(x)$, $Q(x)$, and $R(x)$. The latter equation can be integrated once immediately, resulting in a first order linear equation for y that can be solved as in Section 2.1. By equating the coefficients of the preceding equations and then eliminating $f(x)$, show that a necessary condition for exactness is

$$P''(x) - Q'(x) + R(x) = 0.$$

It can be shown that this is also a sufficient condition.

Solution

Expand the left side of the second ODE.

$$P'(x)y' + P(x)y'' + f'(x)y + f(x)y' = 0$$

$$P(x)y'' + [P'(x) + f(x)]y' + f'(x)y = 0$$

Equate the coefficients of this and the first ODE.

$$\begin{aligned} P'(x) + f(x) &= Q(x) \\ f'(x) &= R(x) \end{aligned}$$

Differentiate both sides of the first equation with respect to x.

$$P''(x) + f'(x) = Q'(x)$$

Substitute $R(x)$ for $f'(x)$.

$$P''(x) + R(x) = Q'(x)$$

Therefore,

$$P''(x) - Q'(x) + R(x) = 0.$$