Problem 45

In each of Problems 42 through 45, use the result of Problem 41 to determine whether the given equation is exact. If it is, then solve the equation.

$$x^2y'' + xy' - y = 0, \quad x > 0$$

Solution

Divide both sides by x^2.

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$$

The last two terms can be written as $[(1/x)y]'$ by the product rule.

$$y'' + \left[\left(\frac{1}{x}\right)y\right]' = 0$$

Integrate both sides with respect to x.

$$y' + \left(\frac{1}{x}\right)y = C_1$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^x \frac{1}{s}\,ds\right) = e^{\ln x} = x$$

Proceed with the multiplication.

$$xy' + y = C_1x$$

The left side can be written as $(xy)'$ by the product rule.

$$(xy)' = C_1x$$

Integrate both sides with respect to x once more.

$$xy = \frac{C_1}{2}x^2 + C_2$$

Therefore, dividing both sides by x and using a new constant C_3 for $C_1/2$,

$$y(x) = C_3x + \frac{C_2}{x}.$$