Problem 48

In each of Problems 47 through 49, use the result of Problem 46 to find the adjoint of the given differential equation.

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \qquad \text{Legendre's equation}$$

Solution

To make the ODE exact, multiply both sides by an integrating factor $\mu = \mu(x)$.

$$(1-x^2)\mu(x)y'' - 2x\mu(x)y' + \alpha(\alpha+1)\mu(x)y = 0 \tag{1}$$

Now that it's exact, it can be written in the form,

$$[(1-x^2)\mu(x)y']' + [f(x)y]' = 0.$$

Expand the left side.

$$-2x\mu(x)y' + (1-x^2)\mu'(x)y' + (1-x^2)\mu(x)y'' + f'(x)y + f(x)y' = 0$$

Factor it now.

$$(1-x^2)\mu(x)y'' + [(1-x^2)\mu'(x) - 2x\mu(x) + f(x)]y' + f'(x)y = 0$$

Equate the coefficients with those of equation (1).

$$\begin{aligned} (1-x^2)\mu'(x) - 2x\mu(x) + f(x) &= -2x\mu(x) \\ f'(x) &= \alpha(\alpha+1)\mu(x) \end{aligned}$$

Add $2x\mu(x)$ to both sides of the first equation.

$$(1-x^2)\mu'(x) + f(x) = 0$$

Differentiate both sides with respect to x.

$$-2x\mu'(x) + (1-x^2)\mu''(x) + f'(x) = 0$$

Substitute $\alpha(\alpha+1)\mu(x)$ for $f'(x)$.

$$-2x\mu'(x) + (1-x^2)\mu''(x) + \alpha(\alpha+1)\mu(x) = 0$$

Therefore,

$$(1-x^2)\mu''(x) - 2x\mu'(x) + \alpha(\alpha+1)\mu(x) = 0.$$