Problem 50

For the second order linear equation $P(x)y'' + Q(x)y' + R(x)y = 0$, show that the adjoint of the adjoint equation is the original equation.

Solution

Suppose we have the second-order ODE,

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

To make it exact, multiply both sides by an integrating factor $\mu = \mu(x)$.

$$\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = 0 \tag{1}$$

Now that it's exact, it can be written in the form,

$$[\mu(x)P(x)y']' + [f(x)y]' = 0.$$

Expand the left side.

$$\mu'(x)P(x)y' + \mu(x)P'(x)y' + \mu(x)P(x)y'' + f'(x)y + f(x)y' = 0$$

Factor it now.

$$\mu(x)P(x)y'' + [\mu'(x)P(x) + \mu(x)P'(x) + f(x)]y' + f'(x)y = 0$$

Equate the coefficients with those of equation (1).

$$\begin{aligned} \mu'(x)P(x) + \mu(x)P'(x) + f(x) &= \mu(x)Q(x) \\ f'(x) &= \mu(x)R(x) \end{aligned}$$

Differentiate both sides of the first equation with respect to x.

$$\mu''(x)P(x) + \mu'(x)P'(x) + \mu'(x)P'(x) + \mu(x)P''(x) + f'(x) = \mu'(x)Q(x) + \mu(x)Q'(x)$$

Substitute $\mu(x)R(x)$ for $f'(x)$.

$$\mu''(x)P(x) + \mu'(x)P'(x) + \mu'(x)P'(x) + \mu(x)P''(x) + \mu(x)R(x) = \mu'(x)Q(x) + \mu(x)Q'(x)$$

Therefore, the adjoint is

$$P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0.$$

To find the adjoint of the adjoint, multiply both sides by an integrating factor $\gamma = \gamma(x)$.

$$P(x)\gamma(x)\mu''(x) + [2P'(x) - Q(x)]\gamma(x)\mu'(x) + [P''(x) - Q'(x) + R(x)]\gamma(x)\mu(x) = 0 \tag{2}$$

Now that it's exact, it can be written in the form,

$$[P(x)\gamma(x)\mu']' + [g(x)\mu]' = 0.$$

Expand the left side.

$$P'(x)\gamma(x)\mu' + P(x)\gamma'(x)\mu' + P(x)\gamma(x)\mu'' + g'(x)\mu + g(x)\mu' = 0$$

Factor it now.

$$P(x)\gamma(x)\mu'' + [P(x)\gamma'(x) + P'(x)\gamma(x) + g(x)]\mu' + g'(x)\mu = 0$$

Equate the coefficients with those of equation (2).

$$\begin{aligned} P(x)\gamma'(x) + P'(x)\gamma(x) + g(x) &= [2P'(x) - Q(x)]\gamma(x) \\ g'(x) &= [P''(x) - Q'(x) + R(x)]\gamma(x) \end{aligned}$$

Differentiate both sides of the first equation with respect to x.

$$P'(x)\gamma'(x) + P(x)\gamma''(x) + P''(x)\gamma(x) + P'(x)\gamma'(x) + g'(x) = [2P''(x) - Q'(x)]\gamma(x) + [2P'(x) - Q(x)]\gamma'(x)$$

Combine like-terms on the left side and expand the right side.

$$P(x)\gamma''(x) + P''(x)\gamma(x) + \cancel{2P'(x)\gamma'(x)} + g'(x) = 2P''(x)\gamma(x) - Q'(x)\gamma(x) + \cancel{2P'(x)\gamma'(x)} - Q(x)\gamma'(x)$$

Substitute $[P''(x) - Q'(x) + R(x)]\gamma(x)$ for $g'(x)$.

$$P(x)\gamma''(x) + P''(x)\gamma(x) + [P''(x) - Q'(x) + R(x)]\gamma(x) = 2P''(x)\gamma(x) - Q'(x)\gamma(x) - Q(x)\gamma'(x)$$

Expand the left side.

$$P(x)\gamma''(x) + P''(x)\gamma(x) + P''(x)\gamma(x) - Q'(x)\gamma(x) + R(x)\gamma(x) = 2P''(x)\gamma(x) - Q'(x)\gamma(x) - Q(x)\gamma'(x)$$

Combine like-terms on the left side and cancel common terms.

$$P(x)\gamma''(x) + \cancel{2P''(x)\gamma(x)} - \cancel{Q'(x)\gamma(x)} + R(x)\gamma(x) = \cancel{2P''(x)\gamma(x)} - \cancel{Q'(x)\gamma(x)} - Q(x)\gamma'(x)$$

Therefore, bringing $Q(x)\gamma'(x)$ to the left side,

$$P(x)\gamma''(x) + Q(x)\gamma'(x) + R(x)\gamma(x) = 0.$$

This is the same ODE we started with.