Problem 51

A second order linear equation $P(x)y'' + Q(x)y' + R(x)y = 0$ is said to be self-adjoint if its adjoint is the same as the original equation. Show that a necessary condition for this equation to be self-adjoint is that $P'(x) = Q(x)$. Determine whether each of the equations in Problems 47 through 49 is self-adjoint.

Solution

Suppose we have the second-order ODE,

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

To make it exact, multiply both sides by an integrating factor $\mu = \mu(x)$.

$$\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = 0 \tag{1}$$

Now that it's exact, it can be written in the form,

$$[\mu(x)P(x)y']' + [f(x)y]' = 0.$$

Expand the left side.

$$\mu'(x)P(x)y' + \mu(x)P'(x)y' + \mu(x)P(x)y'' + f'(x)y + f(x)y' = 0$$

Factor it now.

$$\mu(x)P(x)y'' + [\mu'(x)P(x) + \mu(x)P'(x) + f(x)]y' + f'(x)y = 0$$

Equate the coefficients with those of equation (1).

$$\begin{aligned} \mu'(x)P(x) + \mu(x)P'(x) + f(x) &= \mu(x)Q(x) \\ f'(x) &= \mu(x)R(x) \end{aligned}$$

Differentiate both sides of the first equation with respect to x.

$$\mu''(x)P(x) + \mu'(x)P'(x) + \mu'(x)P'(x) + \mu(x)P''(x) + f'(x) = \mu'(x)Q(x) + \mu(x)Q'(x)$$

Substitute $\mu(x)R(x)$ for $f'(x)$.

$$\mu''(x)P(x) + \mu'(x)P'(x) + \mu'(x)P'(x) + \mu(x)P''(x) + \mu(x)R(x) = \mu'(x)Q(x) + \mu(x)Q'(x)$$

Therefore, the adjoint is

$$P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0.$$

For the original ODE to be self-adjoint, the coefficients must be equal.

$$\begin{aligned} P &= P(x) \\ 2P' - Q &= Q(x) \\ P'' - Q' + R &= R(x) \end{aligned}$$

The second of these equations is the necessary condition for an ODE to be self-adjoint: $P'(x) = Q(x)$. The third equation is $P''(x) = Q'(x)$, a derivative of the second one.

For Bessel's equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

$P(x) = x^2$ and $Q(x) = x$, which means $P'(x) \neq Q(x)$. Bessel's equation is not self-adjoint.

For Legendre's equation,

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

$P(x) = 1 - x^2$ and $Q(x) = -2x$, which means $P'(x) = Q(x)$. Legendre's equation is self-adjoint.

For Airy's equation,

$$y'' - xy = 0,$$

$P(x) = 1$ and $Q(x) = 0$, which means $P'(x) = Q(x)$. Airy's equation is self-adjoint.