Problem 10

In each of Problems 7 through 16, find the general solution of the given differential equation.

$$y'' + 2y' + 2y = 0$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 2(re^{rt}) + 2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 2r + 2 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(2)(1)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$r = \{-1 - i, -1 + i\}$$

Two solutions to the ODE are $y = e^{(-1-i)t}$ and $y = e^{(-1+i)t}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
y(t) &= C_1 e^{(-1-i)t} + C_2 e^{(-1+i)t} \\
&= C_1 e^{-t-it} + C_2 e^{-t+it} \\
&= C_1 e^{-t} e^{-it} + C_2 e^{-t} e^{it} \\
&= C_1 e^{-t}[\cos(-t) + i \sin(-t)] + C_2 e^{-t}(\cos t + i \sin t) \\
&= C_1 e^{-t}(\cos t - i \sin t) + C_2 e^{-t}(\cos t + i \sin t) \\
&= C_1 e^{-t} \cos t - iC_1 e^{-t} \sin t + C_2 e^{-t} \cos t + iC_2 e^{-t} \sin t \\
&= (C_1 + C_2) e^{-t} \cos t + (-iC_1 + iC_2) e^{-t} \sin t
\end{aligned}$$

Therefore, using C_3 for $C_1 + C_2$ and C_4 for $-iC_1 + iC_2$, the real general solution is

$$y(t) = C_3 e^{-t} \cos t + C_4 e^{-t} \sin t.$$