Problem 12

In each of Problems 7 through 16, find the general solution of the given differential equation.

$$4y'' + 9y = 0$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$4(r^2 e^{rt}) + 9(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$4r^2 + 9 = 0$$
$$r^2 = -\frac{9}{4}$$
$$r = \left\{-\frac{3}{2}i, \frac{3}{2}i\right\}$$

Two solutions to the ODE are $y = e^{-3it/2}$ and $y = e^{3it/2}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
y(t) &= C_1 e^{-3it/2} + C_2 e^{3it/2} \\
&= C_1\left[\cos\left(-\frac{3t}{2}\right) + i\sin\left(-\frac{3t}{2}\right)\right] + C_2\left[\cos\left(\frac{3t}{2}\right) + i\sin\left(\frac{3t}{2}\right)\right] \\
&= C_1\left(\cos\frac{3t}{2} - i\sin\frac{3t}{2}\right) + C_2\left(\cos\frac{3t}{2} + i\sin\frac{3t}{2}\right) \\
&= C_1\cos\frac{3t}{2} - iC_1\sin\frac{3t}{2} + C_2\cos\frac{3t}{2} + iC_2\sin\frac{3t}{2} \\
&= (C_1 + C_2)\cos\frac{3t}{2} + (-iC_1 + iC_2)\sin\frac{3t}{2}
\end{aligned}$$

Therefore, using C_3 for $C_1 + C_2$ and C_4 for $-iC_1 + iC_2$, the real general solution is

$$y(t) = C_3\cos\frac{3t}{2} + C_4\sin\frac{3t}{2}.$$