Problem 16

In each of Problems 7 through 16, find the general solution of the given differential equation.

$$y'' + 4y' + 6.25y = 0$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 4(re^{rt}) + 6.25(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 4r + 6.25 = 0$$

$$r = \frac{-4 \pm \sqrt{16 - 4(6.25)(1)}}{2} = \frac{-4 \pm \sqrt{-9}}{2} = \frac{-4 \pm 3i}{2} = -2 \pm \frac{3i}{2}$$

$$r = \left\{-2 - \frac{3i}{2}, -2 + \frac{3i}{2}\right\}$$

Two solutions to the ODE are $y = e^{(-2-3i/2)t}$ and $y = e^{(-2+3i/2)t}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
y(t) &= C_1 e^{(-2-3i/2)t} + C_2 e^{(-2+3i/2)t} \\
&= C_1 e^{-2t-3it/2} + C_2 e^{-2t+3it/2} \\
&= C_1 e^{-2t} e^{-3it/2} + C_2 e^{-2t} e^{3it/2} \\
&= C_1 e^{-2t}\left[\cos\left(-\frac{3t}{2}\right) + i\sin\left(-\frac{3t}{2}\right)\right] + C_2 e^{-2t}\left[\cos\left(\frac{3t}{2}\right) + i\sin\left(\frac{3t}{2}\right)\right] \\
&= C_1 e^{-2t}\left(\cos\frac{3t}{2} - i\sin\frac{3t}{2}\right) + C_2 e^{-2t}\left(\cos\frac{3t}{2} + i\sin\frac{3t}{2}\right) \\
&= C_1 e^{-2t}\cos\frac{3t}{2} - iC_1 e^{-2t}\sin\frac{3t}{2} + C_2 e^{-2t}\cos\frac{3t}{2} + iC_2 e^{-2t}\sin\frac{3t}{2} \\
&= (C_1 + C_2)e^{-2t}\cos\frac{3t}{2} + (-iC_1 + iC_2)e^{-2t}\sin\frac{3t}{2}
\end{aligned}$$

Therefore, using C_3 for $C_1 + C_2$ and C_4 for $-iC_1 + iC_2$, the real general solution is

$$y(t) = C_3 e^{-2t}\cos\frac{3t}{2} + C_4 e^{-2t}\sin\frac{3t}{2}.$$