Problem 31

If e^{rt} is given by Eq. (13), show that

$$\frac{d}{dt}e^{rt} = re^{rt}$$

for any complex number r.

Solution

Suppose that $r = \lambda + i\mu$ is a complex number and that e^{rt} is given by Eq. (13).

$$\begin{aligned} e^{rt} &= e^{(\lambda+i\mu)t} \\ &= e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t \end{aligned} \tag{13}$$

Differentiate both sides with respect to t.

$$\begin{aligned} \frac{d}{dt}e^{rt} &= \frac{d}{dt}(e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t) \\ &= \frac{d}{dt}(e^{\lambda t}\cos\mu t) + \frac{d}{dt}(ie^{\lambda t}\sin\mu t) \\ &= \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t + i\lambda e^{\lambda t}\sin\mu t + i\mu e^{\lambda t}\cos\mu t) \\ &= \lambda e^{\lambda t}(\cos\mu t + i\sin\mu t) + \mu e^{\lambda t}(i\cos\mu t - \sin\mu t) \\ &= \lambda e^{\lambda t}e^{i\mu t} + \mu e^{\lambda t}(i\cos\mu t + i^2\sin\mu t) \\ &= \lambda e^{\lambda t}e^{i\mu t} + i\mu e^{\lambda t}(\cos\mu t + i\sin\mu t) \\ &= \lambda e^{\lambda t}e^{i\mu t} + i\mu e^{\lambda t}e^{i\mu t} \\ &= (\lambda + i\mu)e^{\lambda t}e^{i\mu t} \\ &= re^{\lambda t + i\mu t} \\ &= re^{(\lambda+i\mu)t} \\ &= re^{rt} \end{aligned}$$