Problem 42

In each of Problems 35 through 42, use the method of Problem 34 to solve the given equation for $t > 0$.

$$t^2y'' + 7ty' + 10y = 0$$

Solution

The Hard Way

Make the substitution $x = \ln t$ in the ODE. Then

$$e^x = t \quad \rightarrow \quad e^{2x} = t^2,$$

and the ODE becomes

$$e^{2x}\frac{d^2y}{dt^2} + 7e^x\frac{dy}{dt} + 10y = 0.$$

The aim now is to find what the derivatives are in terms of this new variable by using the chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\left(\frac{1}{t}\right) = \frac{dy}{dx}\left(\frac{1}{e^x}\right) = e^{-x}\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{dx}{dt}\frac{d}{dx}\left(e^{-x}\frac{dy}{dx}\right) = \frac{1}{t}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) = \frac{1}{e^x}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right)$$

Substitute these expressions into the ODE.

$$e^{2x}\frac{1}{e^x}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) + 7e^x\left(e^{-x}\frac{dy}{dx}\right) + 10y = 0$$

$$e^x\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) + 7\frac{dy}{dx} + 10y = 0$$

$$-\frac{dy}{dx} + \frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 10y = 0$$

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 0 \tag{1}$$

The transformed ODE is one with constant coefficients, so its solution is of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad \frac{dy}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y}{dx^2} = r^2e^{rx}$$

Substitute these expressions into equation (1).

$$r^2e^{rx} + 6(re^{rx}) + 10(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 6r + 10 = 0$$

$$r = \frac{-6 \pm \sqrt{36 - 4(1)(10)}}{2} = \frac{-6 \pm \sqrt{-4}}{2} = \frac{-6 \pm 2i}{2} = -3 \pm i$$

$$r = \{-3 - i, -3 + i\}$$

Two solutions to the ODE are $y = e^{(-3-i)x}$ and $y = e^{(-3+i)x}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
y(x) &= C_1 e^{(-3-i)x} + C_2 e^{(-3+i)x} \\
&= C_1 e^{-3x-ix} + C_2 e^{-3x+ix} \\
&= C_1 e^{-3x} e^{-ix} + C_2 e^{-3x} e^{ix} \\
&= C_1 e^{-3x}[\cos(-x) + i\sin(-x)] + C_2 e^{-3x}[\cos(x) + i\sin(x)] \\
&= C_1 e^{-3x}[\cos(x) - i\sin(x)] + C_2 e^{-3x}[\cos(x) + i\sin(x)] \\
&= C_1 e^{-3x}\cos x - iC_1 e^{-3x}\sin x + C_2 e^{-3x}\cos x + iC_2 e^{-3x}\sin x \\
&= (C_1 + C_2)e^{-3x}\cos x + (-iC_1 + iC_2)e^{-3x}\sin x
\end{aligned}$$

Using C_3 for $C_1 + C_2$ and C_4 for $-iC_1 + iC_2$, the real general solution is

$$y(x) = C_3 e^{-3x}\cos x + C_4 e^{-3x}\sin x.$$

Change back to the original variable now.

$$\begin{aligned}
y(t) &= C_3 e^{-3\ln t}\cos(\ln t) + C_4 e^{-3\ln t}\sin(\ln t) \\
&= C_3 e^{\ln t^{-3}}\cos(\ln t) + C_4 e^{\ln t^{-3}}\sin(\ln t)
\end{aligned}$$

Therefore,

$$y(t) = C_3 t^{-3}\cos(\ln t) + C_4 t^{-3}\sin(\ln t).$$

The Easy Way

$$t^2 y'' + 7ty' + 10y = 0$$

Since this is an Euler (or equidimensional) equation, the solution is of the form $y = t^r$.

$$y = t^r \quad \rightarrow \quad y' = rt^{r-1} \quad \rightarrow \quad y'' = r(r-1)t^{r-2}$$

Substitute these expressions into the ODE.

$$t^2[r(r-1)t^{r-2}] + 7t(rt^{r-1}) + 10t^r = 0$$

$$r(r-1)t^r + 7rt^r + 10t^r = 0$$

Divide both sides by t^r.

$$r(r-1) + 7r + 10 = 0$$

$$r^2 + 6r + 10 = 0$$

$$r = \frac{-6 \pm \sqrt{36 - 4(1)(10)}}{2} = \frac{-6 \pm \sqrt{-4}}{2} = \frac{-6 \pm 2i}{2} = -3 \pm i$$

$$r = \{-3 - i, -3 + i\}$$

Two solutions to the ODE are $y = t^{-3-i}$ and $y = t^{-3+i}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
y(t) &= C_5 t^{-3-i} + C_6 t^{-3+i} \\
&= C_5 t^{-3} t^{-i} + C_6 t^{-3} t^{i} \\
&= C_5 t^{-3} e^{\ln t^{-i}} + C_6 t^{-3} e^{\ln t^{i}} \\
&= C_5 t^{-3} e^{(-i)\ln t} + C_6 t^{-3} e^{(i)\ln t} \\
&= C_5 t^{-3}[\cos(-\ln t) + i\sin(-\ln t)] + C_6 t^{-3}[\cos(\ln t) + i\sin(\ln t)] \\
&= C_5 t^{-3}[\cos(\ln t) - i\sin(\ln t)] + C_6 t^{-3}[\cos(\ln t) + i\sin(\ln t)] \\
&= C_5 t^{-3}\cos(\ln t) - iC_5 t^{-3}\sin(\ln t) + C_6 t^{-3}\cos(\ln t) + iC_6 t^{-3}\sin(\ln t) \\
&= (C_5 + C_6) t^{-3}\cos(\ln t) + (-iC_5 + iC_6) t^{-3}\sin(\ln t)
\end{aligned}$$

Using C_7 for $C_5 + C_6$ and C_8 for $-iC_5 + iC_6$, the real general solution is

$$y(t) = C_7 t^{-3}\cos(\ln t) + C_8 t^{-3}\sin(\ln t).$$