Problem 45

In each of Problems 44 through 46, try to transform the given equation into one with constant coefficients by the method of Problem 43. If this is possible, find the general solution of the given equation.

$$y'' + 3ty' + t^2 y = 0, \qquad -\infty < t < \infty$$

Solution

To turn this ODE into one with constant coefficients, make the change of variables,

$$x = \int^t (s^2)^{1/2}\, ds = \int^t s\, ds = \frac{t^2}{2}.$$

Use the chain rule to write y' and y'' in terms of this new variable.

$$\begin{aligned}
\frac{dy}{dt} &= \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}t \\
\frac{d^2y}{dt^2} &= \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\frac{dx}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dx}{dt} + \frac{dy}{dx}\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dx}{dt}\frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{dt} + \frac{dy}{dx}\frac{d^2x}{dt^2} \\
&= \frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{dy}{dx}\frac{d^2x}{dt^2} = \frac{d^2y}{dx^2}(t)^2 + \frac{dy}{dx}(1) = \frac{d^2y}{dx^2}t^2 + \frac{dy}{dx}
\end{aligned}$$

Substitute these expressions for the derivatives into the ODE.

$$\left(\frac{d^2y}{dx^2}t^2 + \frac{dy}{dx}\right) + 3t\left(\frac{dy}{dx}t\right) + t^2 y = 0$$

$$\frac{d^2y}{dx^2}t^2 + \frac{dy}{dx} + 3t^2\frac{dy}{dx} + t^2 y = 0$$

$$\frac{d^2y}{dx^2}t^2 + (1 + 3t^2)\frac{dy}{dx} + t^2 y = 0$$

The original ODE cannot be changed to one with constant coefficients by the method of Problem 43.