Problem 19

In each of Problems 17 through 22, find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

$$y'' - 2y' + 5y = 0, \qquad y(\pi/2) = 0, \quad y'(\pi/2) = 2$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - 2(re^{rt}) + 5(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - 2r + 5 = 0$$

$$r = \frac{2 \pm \sqrt{4 - 4(5)(1)}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$$r = \{1 - 2i, 1 + 2i\}$$

Two solutions to the ODE are $y = e^{(1-2i)t}$ and $y = e^{(1+2i)t}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
y(t) &= C_1 e^{(1-2i)t} + C_2 e^{(1+2i)t} \\
&= C_1 e^{t-2it} + C_2 e^{t+2it} \\
&= C_1 e^t e^{-2it} + C_2 e^t e^{2it} \\
&= C_1 e^t[\cos(-2t) + i\sin(-2t)] + C_2 e^t[\cos(2t) + i\sin(2t)] \\
&= C_1 e^t(\cos 2t - i\sin 2t) + C_2 e^t(\cos 2t + i\sin 2t) \\
&= C_1 e^t \cos 2t - iC_1 e^t \sin 2t + C_2 e^t \cos 2t + iC_2 e^t \sin 2t \\
&= (C_1 + C_2)e^t \cos 2t + (-iC_1 + iC_2)e^t \sin 2t
\end{aligned}$$

Using C_3 for $C_1 + C_2$ and C_4 for $-iC_1 + iC_2$, the real general solution is

$$y(t) = C_3 e^t \cos 2t + C_4 e^t \sin 2t.$$

Take a derivative of it.

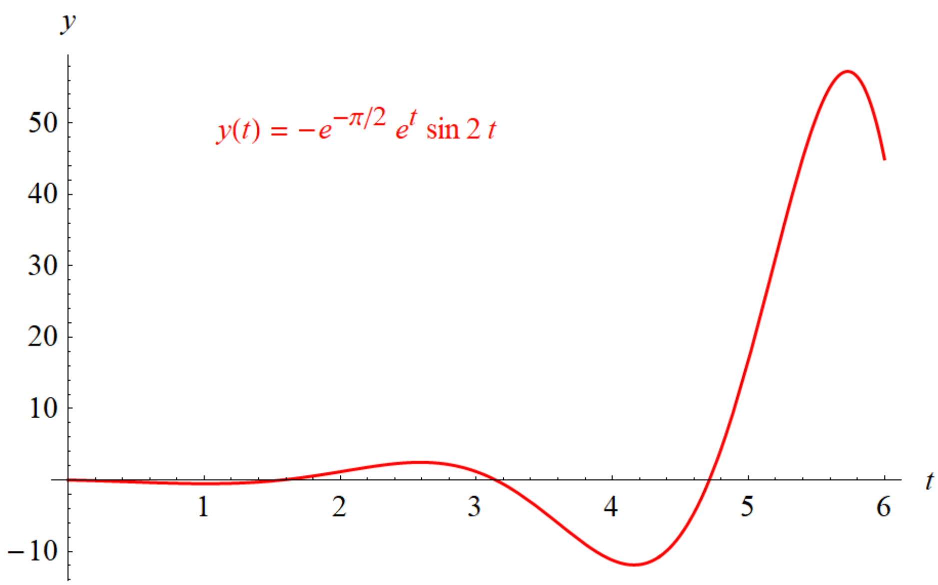
$$y'(t) = C_3 e^t \cos 2t - 2C_3 e^t \sin 2t + C_4 e^t \sin 2t + 2C_4 e^t \cos 2t$$

Apply the initial conditions now to determine C_3 and C_4.

$$\begin{aligned}
y\left(\frac{\pi}{2}\right) &= -C_3 e^{\pi/2} = 0 & \rightarrow \qquad & C_3 = 0 \\
y'\left(\frac{\pi}{2}\right) &= -C_3 e^{\pi/2} - 2C_4 e^{\pi/2} = 2 & \rightarrow \qquad & C_4 = -e^{-\pi/2}
\end{aligned}$$

Therefore,

$$y(t) = -e^{-\pi/2} e^t \sin 2t.$$



The solution oscillates and has an amplitude that exponentially grows.