Problem 22

In each of Problems 17 through 22, find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

$$y'' + 2y' + 2y = 0, \qquad y(\pi/4) = 2, \quad y'(\pi/4) = -2$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 2(re^{rt}) + 2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 2r + 2 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(2)(1)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$r = \{-1 - i, -1 + i\}$$

Two solutions to the ODE are $y = e^{(-1-i)t}$ and $y = e^{(-1+i)t}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
y(t) &= C_1 e^{(-1-i)t} + C_2 e^{(-1+i)t} \\
&= C_1 e^{-t-it} + C_2 e^{-t+it} \\
&= C_1 e^{-t} e^{-it} + C_2 e^{-t} e^{it} \\
&= C_1 e^{-t}[\cos(-t) + i\sin(-t)] + C_2 e^{-t}(\cos t + i \sin t) \\
&= C_1 e^{-t}(\cos t - i \sin t) + C_2 e^{-t}(\cos t + i \sin t) \\
&= C_1 e^{-t} \cos t - iC_1 e^{-t} \sin t + C_2 e^{-t} \cos t + iC_2 e^{-t} \sin t \\
&= (C_1 + C_2) e^{-t} \cos t + (-iC_1 + iC_2) e^{-t} \sin t
\end{aligned}$$

Using C_3 for $C_1 + C_2$ and C_4 for $-iC_1 + iC_2$, the real general solution is

$$y(t) = C_3 e^{-t} \cos t + C_4 e^{-t} \sin t.$$

Take a derivative of it.

$$y'(t) = -C_3 e^{-t} \cos t - C_3 e^{-t} \sin t - C_4 e^{-t} \sin t + C_4 e^{-t} \cos t$$

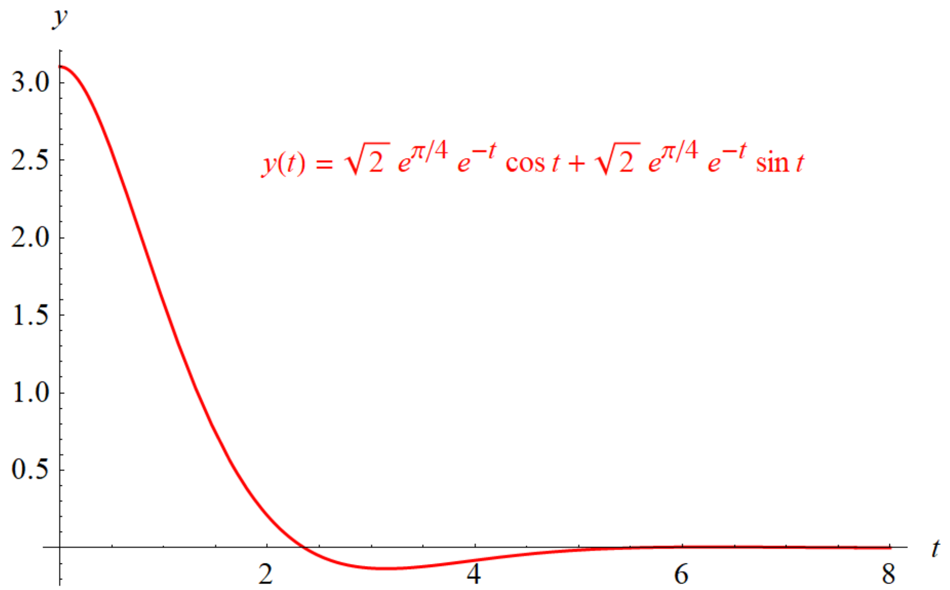
Apply the initial conditions now to determine C_3 and C_4.

$$y\left(\frac{\pi}{4}\right) = C_3 e^{-\pi/4} \left(\frac{\sqrt{2}}{2}\right) + C_4 e^{-\pi/4} \left(\frac{\sqrt{2}}{2}\right) = 2$$

$$y'\left(\frac{\pi}{4}\right) = -C_3 e^{-\pi/4} \left(\frac{\sqrt{2}}{2}\right) - C_3 e^{-\pi/4} \left(\frac{\sqrt{2}}{2}\right) - \cancel{C_4 e^{-\pi/4} \left(\frac{\sqrt{2}}{2}\right)} + \cancel{C_4 e^{-\pi/4} \left(\frac{\sqrt{2}}{2}\right)} = -2$$

Solving this system of equations yields $C_3 = \sqrt{2} e^{\pi/4}$ and $C_4 = \sqrt{2} e^{\pi/4}$. Therefore,

$$y(t) = \sqrt{2} e^{\pi/4} e^{-t} \cos t + \sqrt{2} e^{\pi/4} e^{-t} \sin t.$$



The solution oscillates and has an amplitude that exponentially decays.