Problem 24

Consider the initial value problem

$$5u'' + 2u' + 7u = 0, \qquad u(0) = 2, \quad u'(0) = 1.$$

(a) Find the solution $u(t)$ of this problem.

(b) Find the smallest T such that $|u(t)| \leq 0.1$ for all $t > T$.

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $u = e^{rt}$.

$$u = e^{rt} \quad \rightarrow \quad u' = re^{rt} \quad \rightarrow \quad u'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$5(r^2e^{rt}) + 2(re^{rt}) + 7(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$5r^2 + 2r + 7 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(5)(7)}}{2(5)} = \frac{-2 \pm \sqrt{-136}}{10} = \frac{-2 \pm 2i\sqrt{34}}{10} = -\frac{1}{5} \pm i\frac{\sqrt{34}}{5}$$

$$r = \left\{-\frac{1}{5} - i\frac{\sqrt{34}}{5}, -\frac{1}{5} + i\frac{\sqrt{34}}{5}\right\}$$

Two solutions to the ODE are $u = e^{(-1/5 - i\sqrt{34}/5)t}$ and $u = e^{(-1/5 + i\sqrt{34}/5)t}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
u(t) &= C_1 e^{(-1/5 - i\sqrt{34}/5)t} + C_2 e^{(-1/5 + i\sqrt{34}/5)t} \\
&= C_1 e^{-t/5 - it\sqrt{34}/5} + C_2 e^{-t/5 + it\sqrt{34}/5} \\
&= C_1 e^{-t/5} e^{-it\sqrt{34}/5} + C_2 e^{-t/5} e^{it\sqrt{34}/5} \\
&= C_1 e^{-t/5}\left[\cos\left(-\frac{\sqrt{34}}{5}t\right) + i\sin\left(-\frac{\sqrt{34}}{5}t\right)\right] + C_2 e^{-t/5}\left[\cos\left(\frac{\sqrt{34}}{5}t\right) + i\sin\left(\frac{\sqrt{34}}{5}t\right)\right] \\
&= C_1 e^{-t/5}\left[\cos\left(\frac{\sqrt{34}}{5}t\right) - i\sin\left(\frac{\sqrt{34}}{5}t\right)\right] + C_2 e^{-t/5}\left[\cos\left(\frac{\sqrt{34}}{5}t\right) + i\sin\left(\frac{\sqrt{34}}{5}t\right)\right] \\
&= C_1 e^{-t/5}\cos\left(\frac{\sqrt{34}}{5}t\right) - iC_1 e^{-t/5}\sin\left(\frac{\sqrt{34}}{5}t\right) + C_2 e^{-t/5}\cos\left(\frac{\sqrt{34}}{5}t\right) + iC_2 e^{-t/5}\sin\left(\frac{\sqrt{34}}{5}t\right) \\
&= (C_1 + C_2)e^{-t/5}\cos\left(\frac{\sqrt{34}}{5}t\right) + (-iC_1 + iC_2)e^{-t/5}\sin\left(\frac{\sqrt{34}}{5}t\right)
\end{aligned}$$

Using C_3 for $C_1 + C_2$ and C_4 for $-iC_1 + iC_2$, the real general solution is

$$u(t) = C_3 e^{-t/5}\cos\left(\frac{\sqrt{34}}{5}t\right) + C_4 e^{-t/5}\sin\left(\frac{\sqrt{34}}{5}t\right).$$

Take a derivative of it.

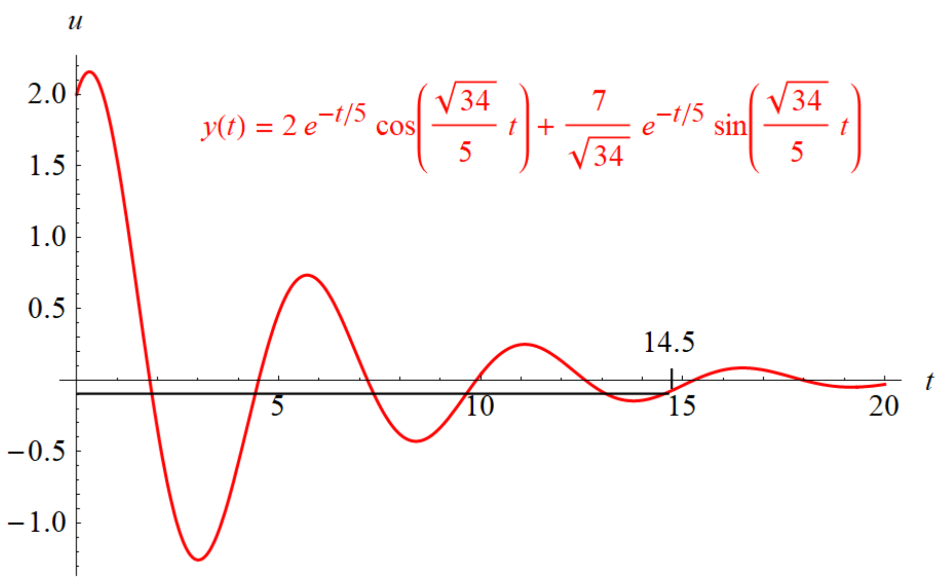
$$u'(t) = -\frac{C_3}{5}e^{-t/5}\cos\left(\frac{\sqrt{34}}{5}t\right) - C_3\frac{\sqrt{34}}{5}e^{-t/5}\sin\left(\frac{\sqrt{34}}{5}t\right) - \frac{C_4}{5}e^{-t/5}\sin\left(\frac{\sqrt{34}}{5}t\right) + C_4\frac{\sqrt{34}}{5}e^{-t/5}\cos\left(\frac{\sqrt{34}}{5}t\right)$$

Apply the initial conditions now to determine C_3 and C_4.

$$u(0) = C_3 = 2$$
$$u'(0) = -\frac{C_3}{5} + C_4\frac{\sqrt{34}}{5} = 1$$

Solving this system of equations yields $C_3 = 2$ and $C_4 = 7/\sqrt{34}$. Therefore,

$$u(t) = 2e^{-t/5}\cos\left(\frac{\sqrt{34}}{5}t\right) + \frac{7}{\sqrt{34}}e^{-t/5}\sin\left(\frac{\sqrt{34}}{5}t\right).$$



Based on the graph, the smallest time for which the amplitude is 0.1 or less is $t \approx 14.5$.