Problem 27

Show that $W(e^{\lambda t}\cos\mu t, e^{\lambda t}\sin\mu t) = \mu e^{2\lambda t}$.

Solution

The Wronskian of $e^{\lambda t}\cos\mu t$ and $e^{\lambda t}\sin\mu t$ is

$$\begin{aligned}
W(e^{\lambda t}\cos\mu t, e^{\lambda t}\sin\mu t) &= \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ \frac{d}{dt}(e^{\lambda t}\cos\mu t) & \frac{d}{dt}(e^{\lambda t}\sin\mu t) \end{vmatrix} \\
&= \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t & \lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t \end{vmatrix} \\
&= e^{\lambda t}\cos\mu t(\lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t) - e^{\lambda t}\sin\mu t(\lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t) \\
&= \cancel{\lambda e^{2\lambda t}\cos\mu t\sin\mu t} + \mu e^{2\lambda t}\cos^2\mu t - \cancel{\lambda e^{2\lambda t}\cos\mu t\sin\mu t} + \mu e^{2\lambda t}\sin^2\mu t \\
&= \mu e^{2\lambda t}(\cos^2\mu t + \sin^2\mu t) \\
&= \mu e^{2\lambda t}.
\end{aligned}$$