Problem 30

If e^{rt} is given by Eq. (13), show that $e^{(r_1+r_2)t} = e^{r_1 t}e^{r_2 t}$ for any complex numbers r_1 and r_2.

Solution

Suppose that $r = \lambda + i\mu$ is a complex number and that e^{rt} is given by Eq. (13).

$$\begin{aligned} e^{rt} &= e^{(\lambda+i\mu)t} \\ &= e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t \end{aligned} \tag{13}$$

Let $r_1 = \lambda_1 + i\mu_1$ and $r_2 = \lambda_2 + i\mu_2$. Then

$$\begin{aligned} e^{(r_1+r_2)t} &= e^{(\lambda_1+i\mu_1+\lambda_2+i\mu_2)t} \\ &= e^{[(\lambda_1+\lambda_2)+i(\mu_1+\mu_2)]t} \\ &= e^{(\lambda_1+\lambda_2)t}\cos[(\mu_1+\mu_2)t] + ie^{(\lambda_1+\lambda_2)t}\sin[(\mu_1+\mu_2)t] \\ &= e^{\lambda_1 t+\lambda_2 t}\cos(\mu_1 t+\mu_2 t) + ie^{\lambda_1 t+\lambda_2 t}\sin(\mu_1 t+\mu_2 t) \\ &= e^{\lambda_1 t}e^{\lambda_2 t}(\cos\mu_1 t\cos\mu_2 t - \sin\mu_1 t\sin\mu_2 t) + ie^{\lambda_1 t}e^{\lambda_2 t}(\sin\mu_1 t\cos\mu_2 t + \cos\mu_1 t\sin\mu_2 t) \\ &= e^{\lambda_1 t}e^{\lambda_2 t}\cos\mu_1 t\cos\mu_2 t - e^{\lambda_1 t}e^{\lambda_2 t}\sin\mu_1 t\sin\mu_2 t \\ &\qquad + ie^{\lambda_1 t}e^{\lambda_2 t}\sin\mu_1 t\cos\mu_2 t + ie^{\lambda_1 t}e^{\lambda_2 t}\cos\mu_1 t\sin\mu_2 t \\ &= e^{\lambda_1 t}e^{\lambda_2 t}\cos\mu_1 t\cos\mu_2 t + i^2 e^{\lambda_1 t}e^{\lambda_2 t}\sin\mu_1 t\sin\mu_2 t \\ &\qquad + ie^{\lambda_1 t}e^{\lambda_2 t}\sin\mu_1 t\cos\mu_2 t + ie^{\lambda_1 t}e^{\lambda_2 t}\cos\mu_1 t\sin\mu_2 t \\ &= (e^{\lambda_1 t}\cos\mu_1 t + ie^{\lambda_1 t}\sin\mu_1 t)(e^{\lambda_2 t}\cos\mu_2 t + ie^{\lambda_2 t}\sin\mu_2 t) \\ &= e^{(\lambda_1+i\mu_1)t}e^{(\lambda_2+i\mu_2)t} \\ &= e^{r_1 t}e^{r_2 t}. \end{aligned}$$