Problem 35

In each of Problems 35 through 42, use the method of Problem 34 to solve the given equation for $t > 0$.

$$t^2y'' + ty' + y = 0$$

Solution

The Hard Way

Make the substitution $x = \ln t$ in the ODE. Then

$$e^x = t \quad \rightarrow \quad e^{2x} = t^2,$$

and the ODE becomes

$$e^{2x}\frac{d^2y}{dt^2} + e^x\frac{dy}{dt} + y = 0.$$

The aim now is to find what the derivatives are in terms of this new variable by using the chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\left(\frac{1}{t}\right) = \frac{dy}{dx}\left(\frac{1}{e^x}\right) = e^{-x}\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{dx}{dt}\frac{d}{dx}\left(e^{-x}\frac{dy}{dx}\right) = \frac{1}{t}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) = \frac{1}{e^x}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right)$$

Substitute these expressions into the ODE.

$$e^{2x}\frac{1}{e^x}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) + e^x\left(e^{-x}\frac{dy}{dx}\right) + y = 0$$

$$e^x\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) + \frac{dy}{dx} + y = 0$$

$$-\cancel{\frac{dy}{dx}} + \frac{d^2y}{dx^2} + \cancel{\frac{dy}{dx}} + y = 0$$

$$\frac{d^2y}{dx^2} + y = 0 \tag{1}$$

The transformed ODE is one with constant coefficients, so its solution is of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad \frac{dy}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y}{dx^2} = r^2e^{rx}$$

Substitute these expressions into equation (1).

$$r^2e^{rx} + e^{rx} = 0$$

Divide both sides by e^{rx}.

$$r^2 + 1 = 0$$

$$r = \{-i, i\}$$

Two solutions to the ODE are $y = e^{-ix}$ and $y = e^{ix}$, so the general solution is a linear combination of the two.

$$\begin{aligned} y(x) &= C_1 e^{-ix} + C_2 e^{ix} \\ &= C_1[\cos(-x) + i\sin(-x)] + C_2[\cos(x) + i\sin(x)] \\ &= C_1(\cos x - i\sin x) + C_2(\cos x + i\sin x) \\ &= C_1\cos x - iC_1\sin x + C_2\cos x + iC_2\sin x \\ &= (C_1 + C_2)\cos x + (-iC_1 + iC_2)\sin x \end{aligned}$$

Using C_3 for $C_1 + C_2$ and C_4 for $-iC_1 + iC_2$, the real general solution is

$$y(x) = C_3\cos x + C_4\sin x.$$

Therefore, changing back to the original variable,

$$y(t) = C_3\cos(\ln t) + C_4\sin(\ln t).$$

The Easy Way

$$t^2y'' + ty' + y = 0$$

Since this is an Euler (or equidimensional) equation, the solution is of the form $y = t^r$.

$$y = t^r \quad \rightarrow \quad y' = rt^{r-1} \quad \rightarrow \quad y'' = r(r-1)t^{r-2}$$

Substitute these expressions into the ODE.

$$t^2[r(r-1)t^{r-2}] + t(rt^{r-1}) + t^r = 0$$

$$r(r-1)t^r + rt^r + t^r = 0$$

Divide both sides by t^r.

$$\begin{aligned} r(r-1) + r + 1 &= 0 \\ r^2 + 1 &= 0 \\ r &= \{-i, i\} \end{aligned}$$

Two solutions to the ODE are $y = t^{-i}$ and $y = t^{-i}$, so the general solution is a linear combination of the two.

$$\begin{aligned} y(t) &= C_5t^{-i} + C_6t^{i} \\ &= C_5e^{\ln t^{-i}} + C_6e^{\ln t^{i}} \\ &= C_5e^{-i\ln t} + C_6e^{i\ln t} \\ &= C_5[\cos(-\ln t) + i\sin(-\ln t)] + C_6[\cos(\ln t) + i\sin(\ln t)] \\ &= C_5[\cos(\ln t) - i\sin(\ln t)] + C_6[\cos(\ln t) + i\sin(\ln t)] \\ &= C_5\cos(\ln t) - iC_5\sin(\ln t) + C_6\cos(\ln t) + iC_6\sin(\ln t) \\ &= (C_5 + C_6)\cos(\ln t) + (-iC_5 + iC_6)\sin(\ln t) \end{aligned}$$

Using C_7 for $C_5 + C_6$ and C_8 for $-iC_5 + iC_6$, the real general solution is

$$y(t) = C_7\cos(\ln t) + C_8\sin(\ln t).$$