Problem 37

In each of Problems 35 through 42, use the method of Problem 34 to solve the given equation for $t > 0$.

$$t^2 y'' + 3ty' + 1.25y = 0$$

Solution

The Hard Way

Make the substitution $x = \ln t$ in the ODE. Then

$$e^x = t \quad \rightarrow \quad e^{2x} = t^2,$$

and the ODE becomes

$$e^{2x}\frac{d^2y}{dt^2} + 3e^x\frac{dy}{dt} + 1.25y = 0.$$

The aim now is to find what the derivatives are in terms of this new variable by using the chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\left(\frac{1}{t}\right) = \frac{dy}{dx}\left(\frac{1}{e^x}\right) = e^{-x}\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{dx}{dt}\frac{d}{dx}\left(e^{-x}\frac{dy}{dx}\right) = \frac{1}{t}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) = \frac{1}{e^x}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right)$$

Substitute these expressions into the ODE.

$$e^{2x}\frac{1}{e^x}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) + 3e^x\left(e^{-x}\frac{dy}{dx}\right) + 1.25y = 0$$

$$e^x\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) + 3\frac{dy}{dx} + 1.25y = 0$$

$$-\frac{dy}{dx} + \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 1.25y = 0$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 1.25y = 0 \tag{1}$$

The transformed ODE is one with constant coefficients, so its solution is of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad \frac{dy}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y}{dx^2} = r^2e^{rx}$$

Substitute these expressions into equation (1).

$$r^2e^{rx} + 2(re^{rx}) + 1.25(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 2r + 1.25 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(1.25)(1)}}{2} = \frac{-2 \pm \sqrt{-1}}{2} = \frac{-2 \pm i}{2} = -1 \pm \frac{i}{2}$$

$$r = \left\{-1 - \frac{i}{2}, -1 + \frac{i}{2}\right\}$$

Two solutions to the ODE are $y = e^{(-1-i/2)x}$ and $y = e^{(-1+i/2)x}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
y(x) &= C_1 e^{(-1-i/2)x} + C_2 e^{(-1+i/2)x} \\
&= C_1 e^{-x-ix/2} + C_2 e^{-x+ix/2} \\
&= C_1 e^{-x} e^{-ix/2} + C_2 e^{-x} e^{ix/2} \\
&= C_1 e^{-x} \left[\cos\left(-\frac{x}{2}\right) + i\sin\left(-\frac{x}{2}\right)\right] + C_2 e^{-x} \left[\cos\left(\frac{x}{2}\right) + i\sin\left(\frac{x}{2}\right)\right] \\
&= C_1 e^{-x} \left[\cos\left(\frac{x}{2}\right) - i\sin\left(\frac{x}{2}\right)\right] + C_2 e^{-x} \left[\cos\left(\frac{x}{2}\right) + i\sin\left(\frac{x}{2}\right)\right] \\
&= C_1 e^{-x} \cos\left(\frac{x}{2}\right) - iC_1 e^{-x} \sin\left(\frac{x}{2}\right) + C_2 e^{-x} \cos\left(\frac{x}{2}\right) + iC_2 e^{-x} \sin\left(\frac{x}{2}\right) \\
&= (C_1 + C_2) e^{-x} \cos\left(\frac{x}{2}\right) + (-iC_1 + iC_2) e^{-x} \sin\left(\frac{x}{2}\right)
\end{aligned}$$

Use the new constants, C_3 and C_4, for $C_1 + C_2$ and $-iC_1 + iC_2$, respectively.

$$y(x) = C_3 e^{-x} \cos\left(\frac{x}{2}\right) + C_4 e^{-x} \sin\left(\frac{x}{2}\right)$$

Change back to the original variable now.

$$\begin{aligned}
y(t) &= C_3 e^{-\ln t} \cos\left(\frac{\ln t}{2}\right) + C_4 e^{-\ln t} \sin\left(\frac{\ln t}{2}\right) \\
&= C_3 e^{\ln t^{-1}} \cos\left(\frac{\ln t}{2}\right) + C_4 e^{\ln t^{-1}} \sin\left(\frac{\ln t}{2}\right)
\end{aligned}$$

Therefore,

$$y(t) = C_3 t^{-1} \cos\left(\frac{\ln t}{2}\right) + C_4 t^{-1} \sin\left(\frac{\ln t}{2}\right).$$

The Easy Way

$$t^2 y'' + 3ty' + 1.25y = 0$$

Since this is an Euler (or equidimensional) equation, the solution is of the form $y = t^r$.

$$y = t^r \quad \rightarrow \quad y' = rt^{r-1} \quad \rightarrow \quad y'' = r(r-1)t^{r-2}$$

Substitute these expressions into the ODE.

$$t^2[r(r-1)t^{r-2}] + 3t(rt^{r-1}) + 1.25t^r = 0$$

$$r(r-1)t^r + 3rt^r + 1.25t^r = 0$$

Divide both sides by t^r.

$$r(r-1) + 3r + 1.25 = 0$$

$$r^2 + 2r + 1.25 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(1.25)(1)}}{2} = \frac{-2 \pm \sqrt{-1}}{2} = \frac{-2 \pm i}{2} = -1 \pm \frac{i}{2}$$

$$r = \left\{-1 - \frac{i}{2}, -1 + \frac{i}{2}\right\}$$

Two solutions to the ODE are $y = t^{-1-i/2}$ and $y = t^{-1+i/2}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
y(t) &= C_5 t^{-1-i/2} + C_6 t^{-1+i/2} \\
&= C_5 t^{-1} t^{-i/2} + C_6 t^{-1} t^{i/2} \\
&= C_5 t^{-1} e^{\ln t^{-i/2}} + C_6 t^{-1} e^{\ln t^{i/2}} \\
&= C_5 t^{-1} e^{(-i/2)\ln t} + C_6 t^{-1} e^{(i/2)\ln t} \\
&= C_5 t^{-1} \left[\cos\left(-\frac{\ln t}{2}\right) + i\sin\left(-\frac{\ln t}{2}\right)\right] + C_6 t^{-1} \left[\cos\left(\frac{\ln t}{2}\right) + i\sin\left(\frac{\ln t}{2}\right)\right] \\
&= C_5 t^{-1} \left[\cos\left(\frac{\ln t}{2}\right) - i\sin\left(\frac{\ln t}{2}\right)\right] + C_6 t^{-1} \left[\cos\left(\frac{\ln t}{2}\right) + i\sin\left(\frac{\ln t}{2}\right)\right] \\
&= C_5 t^{-1} \cos\left(\frac{\ln t}{2}\right) - iC_5 t^{-1} \sin\left(\frac{\ln t}{2}\right) + C_6 t^{-1} \cos\left(\frac{\ln t}{2}\right) + iC_6 t^{-1} \sin\left(\frac{\ln t}{2}\right) \\
&= (C_5 + C_6) t^{-1} \cos\left(\frac{\ln t}{2}\right) + (-iC_5 + iC_6) t^{-1} \sin\left(\frac{\ln t}{2}\right)
\end{aligned}$$

Therefore, using C_7 for $C_5 + C_6$ and C_8 for $-iC_5 + iC_6$, the real general solution is

$$y(t) = C_7 t^{-1} \cos\left(\frac{\ln t}{2}\right) + C_8 t^{-1} \sin\left(\frac{\ln t}{2}\right).$$