

Problem 38

In each of Problems 35 through 42, use the method of Problem 34 to solve the given equation for $t > 0$.

$$t^2 y'' - 4ty' - 6y = 0$$

Solution

The Hard Way

Make the substitution $x = \ln t$ in the ODE. Then

$$e^x = t \quad \rightarrow \quad e^{2x} = t^2,$$

and the ODE becomes

$$e^{2x} \frac{d^2 y}{dt^2} - 4e^x \frac{dy}{dt} - 6y = 0.$$

The aim now is to find what the derivatives are in terms of this new variable by using the chain rule.

$$\begin{aligned} \frac{dy}{dt} &= \frac{dy}{dx} \frac{dx}{dt} = \frac{dy}{dx} \left(\frac{1}{t} \right) = \frac{dy}{dx} \left(\frac{1}{e^x} \right) = e^{-x} \frac{dy}{dx} \\ \frac{d^2 y}{dt^2} &= \frac{d}{dt} \left(\frac{dy}{dt} \right) = \frac{dx}{dt} \frac{d}{dx} \left(e^{-x} \frac{dy}{dx} \right) = \frac{1}{t} \left(-e^{-x} \frac{dy}{dx} + e^{-x} \frac{d^2 y}{dx^2} \right) = \frac{1}{e^x} \left(-e^{-x} \frac{dy}{dx} + e^{-x} \frac{d^2 y}{dx^2} \right) \end{aligned}$$

Substitute these expressions into the ODE.

$$\begin{aligned} e^{2x} \frac{1}{e^x} \left(-e^{-x} \frac{dy}{dx} + e^{-x} \frac{d^2 y}{dx^2} \right) - 4e^x \left(e^{-x} \frac{dy}{dx} \right) - 6y &= 0 \\ e^x \left(-e^{-x} \frac{dy}{dx} + e^{-x} \frac{d^2 y}{dx^2} \right) - 4 \frac{dy}{dx} - 6y &= 0 \\ -\frac{dy}{dx} + \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} - 6y &= 0 \\ \frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} - 6y &= 0 \end{aligned} \tag{1}$$

The transformed ODE is one with constant coefficients, so its solution is of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad \frac{dy}{dx} = r e^{rx} \quad \rightarrow \quad \frac{d^2 y}{dx^2} = r^2 e^{rx}$$

Substitute these expressions into equation (1).

$$r^2 e^{rx} - 5(r e^{rx}) - 6(e^{rx}) = 0$$

Divide both sides by e^{rx} .

$$\begin{aligned} r^2 - 5r - 6 &= 0 \\ (r - 6)(r + 1) &= 0 \\ r &= \{-1, 6\} \end{aligned}$$

Two solutions to the ODE are $y = e^{-x}$ and $y = e^{6x}$, so the general solution is a linear combination of the two.

$$y(x) = C_1 e^{-x} + C_2 e^{6x}$$

Change back to the original variable now.

$$\begin{aligned} y(t) &= C_1 e^{-\ln t} + C_2 e^{6 \ln t} \\ &= C_1 e^{\ln t^{-1}} + C_2 e^{\ln t^6} \end{aligned}$$

Therefore,

$$y(t) = C_1 t^{-1} + C_2 t^6.$$

The Easy Way

$$t^2 y'' - 4t y' - 6y = 0$$

Since this is an Euler (or equidimensional) equation, the solution is of the form $y = t^r$.

$$y = t^r \quad \rightarrow \quad y' = r t^{r-1} \quad \rightarrow \quad y'' = r(r-1)t^{r-2}$$

Substitute these expressions into the ODE.

$$t^2[r(r-1)t^{r-2}] - 4t(rt^{r-1}) - 6t^r = 0$$

$$r(r-1)t^r - 4rt^r - 6t^r = 0$$

Divide both sides by t^r .

$$r(r-1) - 4r - 6 = 0$$

$$r^2 - 5r - 6 = 0$$

$$(r-6)(r+1) = 0$$

$$r = \{-1, 6\}$$

Two solutions to the ODE are $y = t^{-1}$ and $y = t^6$, so the general solution is a linear combination of the two.

$$y(t) = C_3 t^{-1} + C_4 t^6$$