Problem 46

In each of Problems 44 through 46, try to transform the given equation into one with constant coefficients by the method of Problem 43. If this is possible, find the general solution of the given equation.

$$ty'' + (t^2 - 1)y' + t^3y = 0, \qquad 0 < t < \infty$$

Solution

Divide both sides by t.

$$y'' + \left(t - \frac{1}{t}\right)y' + t^2y = 0$$

To turn this ODE into one with constant coefficients, make the change of variables,

$$x = \int^t (s^2)^{1/2}\,ds = \int^t s\,ds = \frac{t^2}{2}.$$

Use the chain rule to write y' and y'' in terms of this new variable.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}t$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\frac{dx}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dx}{dt} + \frac{dy}{dx}\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dx}{dt}\frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{dt} + \frac{dy}{dx}\frac{d^2x}{dt^2}$$

$$= \frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{dy}{dx}\frac{d^2x}{dt^2} = \frac{d^2y}{dx^2}(t)^2 + \frac{dy}{dx}(1) = \frac{d^2y}{dx^2}t^2 + \frac{dy}{dx}$$

Substitute these expressions for the derivatives into the ODE.

$$\left(\frac{d^2y}{dx^2}t^2 + \frac{dy}{dx}\right) + \left(t - \frac{1}{t}\right)\left(\frac{dy}{dx}t\right) + t^2y = 0$$

$$\frac{d^2y}{dx^2}t^2 + \cancel{\frac{dy}{dx}} + (t^2 - \cancel{1})\frac{dy}{dx} + t^2y = 0$$

$$\frac{d^2y}{dx^2}t^2 + t^2\frac{dy}{dx} + t^2y = 0$$

Divide both sides by t^2.

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad \frac{dy}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y}{dx^2} = r^2e^{rx}$$

Substitute these expressions into the ODE.

$$r^2e^{rx} + re^{rx} + e^{rx} = 0$$

Divide both sides by e^{rx}.

$$r^2 + r + 1 = 0$$

$$r = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$r = \left\{-\frac{1}{2} - i\frac{\sqrt{3}}{2}, -\frac{1}{2} + i\frac{\sqrt{3}}{2}\right\}$$

Two solutions to the ODE are $y = e^{(-1/2-i\sqrt{3}/2)x}$ and $y = e^{(-1/2+i\sqrt{3}/2)x}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
y(x) &= C_1 e^{(-1/2-i\sqrt{3}/2)x} + C_2 e^{(-1/2+i\sqrt{3}/2)x} \\
&= C_1 e^{-x/2-i\sqrt{3}x/2} + C_2 e^{-x/2+i\sqrt{3}x/2} \\
&= C_1 e^{-x/2} e^{-i\sqrt{3}x/2} + C_2 e^{-x/2} e^{i\sqrt{3}x/2} \\
&= C_1 e^{-x/2}\left[\cos\left(-\frac{\sqrt{3}}{2}x\right) + i\sin\left(-\frac{\sqrt{3}}{2}x\right)\right] + C_2 e^{-x/2}\left[\cos\left(\frac{\sqrt{3}}{2}x\right) + i\sin\left(\frac{\sqrt{3}}{2}x\right)\right] \\
&= C_1 e^{-x/2}\left[\cos\left(\frac{\sqrt{3}}{2}x\right) - i\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + C_2 e^{-x/2}\left[\cos\left(\frac{\sqrt{3}}{2}x\right) + i\sin\left(\frac{\sqrt{3}}{2}x\right)\right] \\
&= C_1 e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) - iC_1 e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right) + C_2 e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + iC_2 e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right) \\
&= (C_1 + C_2)e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + (-iC_1 + iC_2)e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)
\end{aligned}$$

Using C_3 for $C_1 + C_2$ and C_4 for $-iC_1 + iC_2$, the real general solution is

$$y(x) = C_3 e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + C_4 e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right).$$

Therefore, since $x = t^2/2$,

$$y(t) = C_3 e^{-t^2/4}\cos\left(\frac{\sqrt{3}}{4}t^2\right) + C_4 e^{-t^2/4}\sin\left(\frac{\sqrt{3}}{4}t^2\right).$$