Problem 8

In each of Problems 7 through 16, find the general solution of the given differential equation.

$$y'' - 2y' + 6y = 0$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - 2(re^{rt}) + 6(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - 2r + 6 = 0$$

$$r = \frac{2 \pm \sqrt{4 - 4(6)(1)}}{2} = \frac{2 \pm \sqrt{-20}}{2} = \frac{2 \pm 2i\sqrt{5}}{2} = 1 \pm i\sqrt{5}$$

$$r = \{1 - i\sqrt{5}, 1 + i\sqrt{5}\}$$

Two solutions to the ODE are $y = e^{(1-i\sqrt{5})t}$ and $y = e^{(1+i\sqrt{5})t}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
y(t) &= C_1 e^{(1-i\sqrt{5})t} + C_2 e^{(1+i\sqrt{5})t} \\
&= C_1 e^{t - i\sqrt{5}t} + C_2 e^{t + i\sqrt{5}t} \\
&= C_1 e^t e^{-i\sqrt{5}t} + C_2 e^t e^{i\sqrt{5}t} \\
&= C_1 e^t[\cos(-\sqrt{5}t) + i\sin(-\sqrt{5}t)] + C_2 e^t[\cos(\sqrt{5}t) + i\sin(\sqrt{5}t)] \\
&= C_1 e^t[\cos(\sqrt{5}t) - i\sin(\sqrt{5}t)] + C_2 e^t[\cos(\sqrt{5}t) + i\sin(\sqrt{5}t)] \\
&= C_1 e^t \cos(\sqrt{5}t) - iC_1 e^t \sin(\sqrt{5}t) + C_2 e^t \cos(\sqrt{5}t) + iC_2 e^t \sin(\sqrt{5}t) \\
&= (C_1 + C_2)e^t \cos(\sqrt{5}t) + (-iC_1 + iC_2)e^t \sin(\sqrt{5}t)
\end{aligned}$$

Therefore, using C_3 for $C_1 + C_2$ and C_4 for $-iC_1 + iC_2$, the real general solution is

$$y(t) = C_3 e^t \cos(\sqrt{5}t) + C_4 e^t \sin(\sqrt{5}t).$$