Problem 2

In each of Problems 1 through 10, find the general solution of the given differential equation.

$$9y'' + 6y' + y = 0$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$9(r^2 e^{rt}) + 6(re^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$9r^2 + 6r + 1 = 0$$

$$(3r + 1)^2 = 0$$

$$r = \left\{-\frac{1}{3}\right\}$$

One solution to the ODE is $y = e^{-t/3}$. Because the ODE is homogeneous, any constant multiple of this is also a solution, that is, $y = ce^{-t/3}$. According to the method of reduction of order, the general solution is found by allowing c to vary as a function of t.

$$y(t) = c(t)e^{-t/3}$$

Substitute this expression for y into the original ODE to determine $c(t)$.

$$9y'' + 6y' + y = 0 \quad \rightarrow \quad 9[c(t)e^{-t/3}]'' + 6[c(t)e^{-t/3}]' + [c(t)e^{-t/3}] = 0$$

Evaluate the derivatives using the product rule.

$$9\left[c'(t)e^{-t/3} - \frac{c(t)}{3}e^{-t/3}\right]' + 6\left[c'(t)e^{-t/3} - \frac{c(t)}{3}e^{-t/3}\right] + c(t)e^{-t/3} = 0$$

$$9\left[c''(t)e^{-t/3} - \frac{c'(t)}{3}e^{-t/3} - \frac{c'(t)}{3}e^{-t/3} + \frac{c(t)}{9}e^{-t/3}\right] + 6\left[c'(t)e^{-t/3} - \frac{c(t)}{3}e^{-t/3}\right] + c(t)e^{-t/3} = 0$$

$$9c''(t)e^{-t/3} - \cancel{3c'(t)e^{-t/3}} - \cancel{3c'(t)e^{-t/3}} + \cancel{c(t)e^{-t/3}} + \cancel{6c'(t)e^{-t/3}} - \cancel{2c(t)e^{-t/3}} + \cancel{c(t)e^{-t/3}} = 0$$

$$9c''(t)e^{-t/3} = 0$$

Divide both sides by $9e^{-t/3}$.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1 t + C_2$$

Therefore, the general solution is

$$y(t) = C_1 te^{-t/3} + C_2 e^{-t/3}.$$