Problem 5

In each of Problems 1 through 10, find the general solution of the given differential equation.

$$y'' - 2y' + 10y = 0$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - 2(re^{rt}) + 10(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - 2r + 10 = 0$$

$$r = \frac{2 \pm \sqrt{4 - 4(1)(10)}}{2} = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

$$r = \{1 - 3i, 1 + 3i\}$$

Two solutions to the ODE are $y = e^{(1-3i)t}$ and $y = e^{(1+3i)t}$, so the general solution is a linear combination of the two.

$$\begin{aligned} y(t) &= C_1 e^{(1-3i)t} + C_2 e^{(1+3i)t} \\ &= C_1 e^{t-3it} + C_2 e^{t+3it} \\ &= C_1 e^t e^{-3it} + C_2 e^t e^{3it} \\ &= C_1 e^t [\cos(-3t) + i \sin(-3t)] + C_2 e^t [\cos(3t) + i \sin(3t)] \\ &= C_1 e^t [\cos(3t) - i \sin(3t)] + C_2 e^t [\cos(3t) + i \sin(3t)] \\ &= C_1 e^t \cos 3t - iC_1 e^t \sin 3t + C_2 e^t \cos 3t + iC_2 e^t \sin 3t \\ &= (C_1 + C_2) e^t \cos 3t + (-iC_1 + iC_2) e^t \sin 3t \end{aligned}$$

Therefore, using C_3 for $C_1 + C_2$ and C_4 for $-iC_1 + iC_2$, the real general solution is

$$y(t) = C_3 e^t \cos 3t + C_4 e^t \sin 3t.$$