Problem 11

In each of Problems 11 through 14, solve the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

$$9y'' - 12y' + 4y = 0, \qquad y(0) = 2, \quad y'(0) = -1$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$9(r^2 e^{rt}) - 12(re^{rt}) + 4(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$9r^2 - 12r + 4 = 0$$

$$(3r - 2)^2 = 0$$

$$r = \left\{ \frac{2}{3} \right\}$$

One solution to the ODE is $y = e^{2t/3}$. Because the ODE is homogeneous, any constant multiple of this is also a solution, that is, $y = ce^{2t/3}$. According to the method of reduction of order, the general solution is found by allowing c to vary as a function of t.

$$y(t) = c(t)e^{2t/3}$$

Substitute this expression for y into the original ODE to determine $c(t)$.

$$9y'' - 12y' + 4y = 0 \quad \rightarrow \quad 9[c(t)e^{2t/3}]'' - 12[c(t)e^{2t/3}]' + 4[c(t)e^{2t/3}] = 0$$

Evaluate the derivatives using the product rule.

$$9\left[c'(t)e^{2t/3} + \frac{2}{3}c(t)e^{2t/3}\right]' - 12\left[c'(t)e^{2t/3} + \frac{2}{3}c(t)e^{2t/3}\right] + 4[c(t)e^{2t/3}] = 0$$

$$9\left[c''(t)e^{2t/3} + \frac{2}{3}c'(t)e^{2t/3} + \frac{2}{3}c'(t)e^{2t/3} + \frac{4}{9}c(t)e^{2t/3}\right] - 12\left[c'(t)e^{2t/3} + \frac{2}{3}c(t)e^{2t/3}\right] + 4[c(t)e^{2t/3}] = 0$$

$$9c''(t)e^{2t/3} + \cancel{6c'(t)e^{2t/3}} + \cancel{6c'(t)e^{2t/3}} + \cancel{4c(t)e^{2t/3}} - \cancel{12c'(t)e^{2t/3}} - \cancel{8c(t)e^{2t/3}} + \cancel{4c(t)e^{2t/3}} = 0$$

$$9c''(t)e^{2t/3} = 0$$

Divide both sides by $9e^{2t/3}$.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1 t + C_2$$

The general solution is then

$$y(t) = C_1 t e^{2t/3} + C_2 e^{2t/3}.$$

Differentiate it with respect to t.

$$y'(t) = C_1 e^{2t/3} + \frac{2}{3} C_1 t e^{2t/3} + \frac{2}{3} C_2 e^{2t/3}$$

Apply the initial conditions now to determine C_1 and C_2.

$$\begin{aligned} y(0) &= C_2 = 2 \\ y'(0) &= C_1 + \frac{2}{3} C_2 = -1 \end{aligned}$$

Solving this system of equations yields $C_1 = -7/3$ and $C_2 = 2$. Therefore,

$$y(t) = -\frac{7}{3} t e^{2t/3} + 2e^{2t/3}.$$

Take the limit of $y(t)$ as $t \to \infty$.

$$\begin{aligned} \lim_{t\to\infty} y(t) &= \lim_{t\to\infty} \left(-\frac{7}{3} t e^{2t/3} + 2e^{2t/3} \right) \\ &= \lim_{t\to\infty} \left(-\frac{7}{3} t + 2 \right) e^{2t/3} \\ &= -\infty \end{aligned}$$

Below is a plot of $y(t)$ versus t.