Problem 14

In each of Problems 11 through 14, solve the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

$$y'' + 4y' + 4y = 0, \qquad y(-1) = 2, \quad y'(-1) = 1$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 4(re^{rt}) + 4(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 4r + 4 = 0$$
$$(r+2)^2 = 0$$
$$r = \{-2\}$$

One solution to the ODE is $y = e^{-2t}$. Because the ODE is homogeneous, any constant multiple of this is also a solution, that is, $y = ce^{-2t}$. According to the method of reduction of order, the general solution is found by allowing c to vary as a function of t.

$$y(t) = c(t)e^{-2t}$$

Substitute this expression for y into the original ODE to determine $c(t)$.

$$y'' + 4y' + 4y = 0 \quad \rightarrow \quad [c(t)e^{-2t}]'' + 4[c(t)e^{-2t}]' + 4[c(t)e^{-2t}] = 0$$

Evaluate the derivatives using the product rule.

$$[c'(t)e^{-2t} - 2c(t)e^{-2t}]' + 4[c'(t)e^{-2t} - 2c(t)e^{-2t}] + 4[c(t)e^{-2t}] = 0$$
$$[c''(t)e^{-2t} - 2c'(t)e^{-2t} - 2c'(t)e^{-2t} + 4c(t)e^{-2t}] + 4[c'(t)e^{-2t} - 2c(t)e^{-2t}] + 4[c(t)e^{-2t}] = 0$$
$$c''(t)e^{-2t} - \cancel{2c'(t)e^{-2t}} - \cancel{2c'(t)e^{-2t}} + \cancel{4c(t)e^{-2t}} + \cancel{4c'(t)e^{-2t}} - \cancel{8c(t)e^{-2t}} + \cancel{4c(t)e^{-2t}} = 0$$
$$c''(t)e^{-2t} = 0$$

Divide both sides by e^{-2t}.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1 t + C_2$$

The general solution is then

$$y(t) = C_1 te^{-2t} + C_2 e^{-2t}.$$

Differentiate it with respect to t.

$$y'(t) = C_1 e^{-2t} - 2C_1 t e^{-2t} - 2C_2 e^{-2t}$$

Apply the initial conditions now to determine C_1 and C_2.

$$\begin{aligned} y(-1) &= -C_1 e^2 + C_2 e^2 = 2 \\ y'(-1) &= C_1 e^2 + 2C_1 e^2 - 2C_2 e^2 = 1 \end{aligned}$$

Solving this system of equations yields $C_1 = 5/e^2$ and $C_2 = 7/e^2$. Therefore,

$$y(t) = \frac{5}{e^2} t e^{-2t} + \frac{7}{e^2} e^{-2t}.$$

Take the limit of $y(t)$ as $t \to \infty$.

$$\begin{aligned} \lim_{t\to\infty} y(t) &= \lim_{t\to\infty} \left(\frac{5}{e^2} t e^{-2t} + \frac{7}{e^2} e^{-2t} \right) \\ &= \lim_{t\to\infty} \left(\frac{5}{e^2} t + \frac{7}{e^2} \right) e^{-2t} \\ &= \lim_{t\to\infty} \frac{\frac{5}{e^2} t + \frac{7}{e^2}}{e^{2t}} \\ &\overset{\frac{\infty}{\infty}}{\underset{\text{H}}{=}} \lim_{t\to\infty} \frac{\frac{5}{e^2}}{2e^{2t}} \\ &= 0 \end{aligned}$$

Below is a plot of $y(t)$ versus t.