Problem 17

Consider the initial value problem

$$4y'' + 4y' + y = 0, \qquad y(0) = 1, \quad y'(0) = 2.$$

(a) Solve the initial value problem and plot the solution.

(b) Determine the coordinates (t_M, y_M) of the maximum point.

(c) Change the second initial condition to $y'(0) = b > 0$ and find the solution as a function of b.

(d) Find the coordinates (t_M, y_M) of the maximum point in terms of b. Describe the dependence of t_M and y_M on b as b increases.

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$4(r^2 e^{rt}) + 4(re^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$4r^2 + 4r + 1 = 0$$

$$(2r+1)^2 = 0$$

$$r = \left\{-\frac{1}{2}\right\}$$

One solution to the ODE is $y = e^{-t/2}$. Because the ODE is homogeneous, any constant multiple of this is also a solution, that is, $y = ce^{-t/2}$. According to the method of reduction of order, the general solution is found by allowing c to vary as a function of t.

$$y(t) = c(t)e^{-t/2}$$

Substitute this expression for y into the original ODE to determine $c(t)$.

$$4y'' + 4y' + y = 0 \quad \rightarrow \quad 4[c(t)e^{-t/2}]'' + 4[c(t)e^{-t/2}]' + c(t)e^{-t/2} = 0$$

Evaluate the derivatives using the product rule.

$$4\left[c'(t)e^{-t/2} - \frac{1}{2}c(t)e^{-t/2}\right]' + 4\left[c'(t)e^{-t/2} - \frac{1}{2}c(t)e^{-t/2}\right] + c(t)e^{-t/2} = 0$$

$$4\left[c''(t)e^{-t/2} - \frac{1}{2}c'(t)e^{-t/2} - \frac{1}{2}c'(t)e^{-t/2} + \frac{1}{4}c(t)e^{-t/2}\right] + 4\left[c'(t)e^{-t/2} - \frac{1}{2}c(t)e^{-t/2}\right] + c(t)e^{-t/2} = 0$$

$$4c''(t)e^{-t/2} - \cancel{2c'(t)e^{-t/2}} - \cancel{2c'(t)e^{-t/2}} + \cancel{c(t)e^{-t/2}} + \cancel{4c'(t)e^{-t/2}} - \cancel{2c(t)e^{-t/2}} + \cancel{c(t)e^{-t/2}} = 0$$

$$4c''(t)e^{-t/2} = 0$$

Divide both sides by $4e^{-t/2}$.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1 t + C_2$$

The general solution is then

$$y(t) = C_1 t e^{-t/2} + C_2 e^{-t/2}.$$

Differentiate it with respect to t.

$$y'(t) = C_1 e^{-t/2} - \frac{1}{2} C_1 t e^{-t/2} - \frac{1}{2} C_2 e^{-t/2}$$

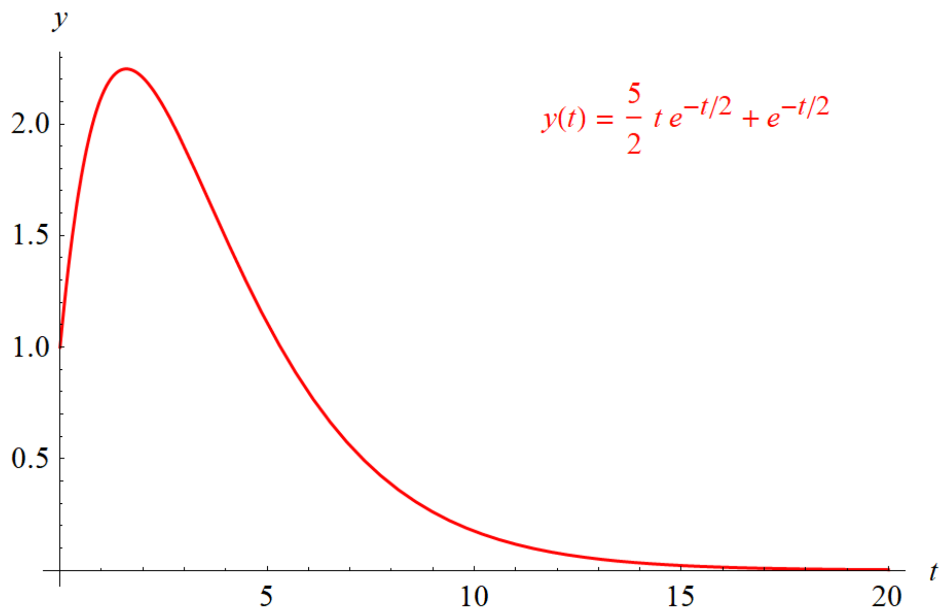
Apply the initial conditions now to determine C_1 and C_2.

$$y(0) = C_2 = 1$$
$$y'(0) = C_1 - \frac{1}{2} C_2 = 2$$

Solving this system of equations yields $C_1 = 5/2$ and $C_2 = 1$. Therefore,

$$y(t) = \frac{5}{2} t e^{-t/2} + e^{-t/2}.$$

Below is a plot of $y(t)$ versus t.



Differentiate the solution with respect to t to get $y'(t)$.

$$y'(t) = \frac{5}{2} e^{-t/2} - \frac{5}{4} t e^{-t/2} - \frac{1}{2} e^{-t/2}$$
$$= -\frac{5}{4} t e^{-t/2} + 2e^{-t/2}$$

Solve $y'(t) = 0$ for t to find the value of t at the maximum.

$$-\frac{5}{4}te^{-t/2} + 2e^{-t/2} = 0$$

$$-\frac{5}{4}t + 2 = 0$$

$$t = \frac{8}{5}$$

The value of y at this value of t is

$$y\left(\frac{8}{5}\right) = \frac{5}{e^{4/5}} \approx 2.25.$$

Therefore, the maximum is at

$$\left(\frac{8}{5}, \frac{5}{e^{4/5}}\right).$$

Return to the general solution and its derivative.

$$y(t) = C_1te^{-t/2} + C_2e^{-t/2}$$
$$y'(t) = C_1e^{-t/2} - \frac{1}{2}C_1te^{-t/2} - \frac{1}{2}C_2e^{-t/2}$$

Apply the new initial conditions now to determine C_1 and C_2.

$$y(0) = C_2 = 1$$
$$y'(0) = C_1 - \frac{1}{2}C_2 = b$$

Solving this system of equations yields $C_1 = b + 1/2$ and $C_2 = 1$. Therefore,

$$y(t) = \left(b + \frac{1}{2}\right)te^{-t/2} + e^{-t/2}.$$

Differentiate the solution with respect to t to get $y'(t)$.

$$y'(t) = \left(b + \frac{1}{2}\right)e^{-t/2} - \frac{1}{2}\left(b + \frac{1}{2}\right)te^{-t/2} - \frac{1}{2}e^{-t/2}$$
$$= -\frac{1}{2}\left(b + \frac{1}{2}\right)te^{-t/2} + be^{-t/2}$$

Solve $y'(t) = 0$ for t to find the value of t at the maximum.

$$-\frac{1}{2}\left(b + \frac{1}{2}\right)te^{-t/2} + be^{-t/2} = 0$$

$$-\frac{1}{2}\left(b + \frac{1}{2}\right)t + b = 0$$

$$t = \frac{2b}{b + \frac{1}{2}} = \frac{4b}{2b + 1}$$

The value of y at this value of t is

$$\begin{aligned} y\left(\frac{4b}{2b+1}\right) &= \left(b+\frac{1}{2}\right)\left(\frac{4b}{2b+1}\right)\exp\left(-\frac{2b}{2b+1}\right) + \exp\left(-\frac{2b}{2b+1}\right) \\ &= \left[\left(b+\frac{1}{2}\right)\left(\frac{2b}{b+\frac{1}{2}}\right)+1\right]\exp\left(-\frac{2b}{2b+1}\right) \\ &= (2b+1)\exp\left(-\frac{2b}{2b+1}\right). \end{aligned}$$

Therefore, the maximum is at

$$\left[\frac{4b}{2b+1}, (2b+1)\exp\left(-\frac{2b}{2b+1}\right)\right].$$

The limit of the t-coordinate as $b \to \infty$ is

$$\begin{aligned} \lim_{b\to\infty} \frac{4b}{2b+1} &= \lim_{b\to\infty} \frac{4}{2+\frac{1}{b}} \\ &= \frac{4}{2} \\ &= 2. \end{aligned}$$

The limit of the y-coordinate as $b \to \infty$ is

$$\begin{aligned} \lim_{b\to\infty} (2b+1)\exp\left(-\frac{2b}{2b+1}\right) &= \lim_{b\to\infty} \frac{2b+1}{\exp\left(\frac{2b}{2b+1}\right)} \\ &= \lim_{b\to\infty} \frac{2b+1}{\exp\left(\frac{2}{2+\frac{1}{b}}\right)} \\ &= \lim_{b\to\infty} \frac{2b+1}{\exp(1)} \\ &= \infty. \end{aligned}$$