Problem 21

Suppose that r_1 and r_2 are roots of $ar^2 + br + c = 0$ and that $r_1 \neq r_2$; then $\exp(r_1 t)$ and $\exp(r_2 t)$ are solutions of the differential equation $ay'' + by' + cy = 0$. Show that $\phi(t; r_1, r_2) = [\exp(r_2 t) - \exp(r_1 t)]/(r_2 - r_1)$ is also a solution of the equation for $r_2 \neq r_1$. Then think of r_1 as fixed, and use L'Hôpital's rule to evaluate the limit of $\phi(t; r_1, r_2)$ as $r_2 \to r_1$, thereby obtaining the second solution in the case of equal roots.

Solution

If $y_1(t) = \exp(r_1 t)$ and $y_2(t) = \exp(r_2 t)$ are solutions to the ODE, then they satisfy

$$\begin{aligned} ay_1'' + by_1' + cy_1 &= 0 \\ ay_2'' + by_2' + cy_2 &= 0. \end{aligned}$$

Divide both sides of each equation by $r_2 - r_1$.

$$\begin{aligned} \frac{a}{r_2 - r_1} y_1'' + \frac{b}{r_2 - r_1} y_1' + \frac{c}{r_2 - r_1} y_1 &= 0 \\ \frac{a}{r_2 - r_1} y_2'' + \frac{b}{r_2 - r_1} y_2' + \frac{c}{r_2 - r_1} y_2 &= 0. \end{aligned}$$

Subtract the respective sides of the first equation from those of the second equation.

$$\frac{a}{r_2 - r_1} y_2'' - \frac{a}{r_2 - r_1} y_1'' + \frac{b}{r_2 - r_1} y_2' - \frac{b}{r_2 - r_1} y_1' + \frac{c}{r_2 - r_1} y_2 - \frac{c}{r_2 - r_1} y_1 = 0$$

Factor a, b, and c.

$$a\left(\frac{1}{r_2 - r_1} y_2'' - \frac{1}{r_2 - r_1} y_1''\right) + b\left(\frac{1}{r_2 - r_1} y_2' - \frac{1}{r_2 - r_1} y_1'\right) + c\left(\frac{1}{r_2 - r_1} y_2 - \frac{1}{r_2 - r_1} y_1\right) = 0$$

Since $1/(r_2 - r_1)$ is a constant, it can be brought inside each of the derivatives.

$$a\left[\left(\frac{y_2}{r_2 - r_1}\right)'' - \left(\frac{y_1}{r_2 - r_1}\right)''\right] + b\left[\left(\frac{y_2}{r_2 - r_1}\right)' - \left(\frac{y_1}{r_2 - r_1}\right)'\right] + c\left(\frac{y_2}{r_2 - r_1} - \frac{y_1}{r_2 - r_1}\right) = 0$$

The derivative is a linear operator.

$$a\left(\frac{y_2}{r_2 - r_1} - \frac{y_1}{r_2 - r_1}\right)'' + b\left(\frac{y_2}{r_2 - r_1} - \frac{y_1}{r_2 - r_1}\right)' + c\left(\frac{y_2}{r_2 - r_1} - \frac{y_1}{r_2 - r_1}\right) = 0$$

$$a\left(\frac{y_2 - y_1}{r_2 - r_1}\right)'' + b\left(\frac{y_2 - y_1}{r_2 - r_1}\right)' + c\left(\frac{y_2 - y_1}{r_2 - r_1}\right) = 0$$

Therefore,

$$\phi(t; r_1, r_2) = \frac{y_2 - y_1}{r_2 - r_1} = \frac{\exp(r_2 t) - \exp(r_1 t)}{r_2 - r_1}$$

also satisfies the ODE.

Take the limit of $\phi(t; r_1, r_2)$ as $r_2 \to r_1$ now.

$$\begin{aligned}
\lim_{r_2 \to r_1} \phi(t; r_1, r_2) &= \lim_{r_2 \to r_1} \frac{\exp(r_2 t) - \exp(r_1 t)}{r_2 - r_1} \\
&\overset{\frac{0}{0}}{\underset{\text{H}}{=}} \lim_{r_2 \to r_1} \frac{\frac{\partial}{\partial r_2}[\exp(r_2 t) - \exp(r_1 t)]}{\frac{\partial}{\partial r_2}(r_2 - r_1)} \\
&= \lim_{r_2 \to r_1} \frac{t e^{r_2 t}}{1} \\
&= t e^{r_1 t}
\end{aligned}$$

In conclusion, if both roots are the same $(r_1 = r_2)$ and one solution is $y = e^{r_1 t}$, then the second solution is $y = t e^{r_1 t}$.