Problem 22

(a) If $ar^2 + br + c = 0$ has equal roots r_1, show that

$$L[e^{rt}] = a(e^{rt})'' + b(e^{rt})' + ce^{rt} = a(r - r_1)^2 e^{rt}. \tag{i}$$

Since the right side of Eq. (i) is zero when $r = r_1$, it follows that $\exp(r_1 t)$ is a solution of $L[y] = ay'' + by' + cy = 0$.

(b) Differentiate Eq. (i) with respect to r, and interchange differentiation with respect to r and with respect to t, thus showing that

$$\frac{\partial}{\partial r} L[e^{rt}] = L\left[\frac{\partial}{\partial r} e^{rt}\right] = L[te^{rt}] = ate^{rt}(r - r_1)^2 + 2ae^{rt}(r - r_1). \tag{ii}$$

Since the right side of Eq. (ii) is zero when $r = r_1$, conclude that $t\exp(r_1 t)$ is also a solution of $L[y] = 0$.

Solution

Part (a)

$$\begin{aligned} L[e^{rt}] &= a(e^{rt})'' + b(e^{rt})' + ce^{rt} \\ &= a(r^2 e^{rt}) + b(re^{rt}) + ce^{rt} \\ &= ar^2 e^{rt} + bre^{rt} + ce^{rt} \\ &= (ar^2 + br + c)e^{rt} \end{aligned}$$

Because r_1 is a root of $ar^2 + br + c = 0$, $ar_1^2 + br_1 + c = 0$, or $c = -ar_1^2 - br_1$.

$$\begin{aligned} &= (ar^2 + br - ar_1^2 - br_1)e^{rt} \\ &= [a(r^2 - r_1^2) + b(r - r_1)]e^{rt} \\ &= [a(r - r_1)(r + r_1) + b(r - r_1)]e^{rt} \\ &= [a(r + r_1) + b](r - r_1)e^{rt} \\ &= a\left(r + r_1 + \frac{b}{a}\right)(r - r_1)e^{rt} \end{aligned}$$

Since $ar^2 + br + c = 0$ has equal roots r_1,

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} = r_1 \quad \rightarrow \quad \frac{b}{a} = -2r_1.$$

Therefore,

$$\begin{aligned} L[e^{rt}] &= a(r + r_1 - 2r_1)(r - r_1)e^{rt} \\ &= a(r - r_1)(r - r_1)e^{rt} \\ &= a(r - r_1)^2 e^{rt}. \end{aligned}$$

Part (b)

Differentiate all sides of Eq. (i) with respect to r.

$$\frac{\partial}{\partial r}L[e^{rt}] = \frac{\partial}{\partial r}\left[a\frac{\partial^2}{\partial t^2}(e^{rt}) + b\frac{\partial}{\partial t}(e^{rt}) + ce^{rt}\right] = \frac{\partial}{\partial r}a(r-r_1)^2e^{rt}$$

Distribute the derivative operator and bring the constants in front.

$$\frac{\partial}{\partial r}L[e^{rt}] = a\frac{\partial}{\partial r}\frac{\partial^2}{\partial t^2}(e^{rt}) + b\frac{\partial}{\partial r}\frac{\partial}{\partial t}(e^{rt}) + c\frac{\partial}{\partial r}e^{rt} = a\frac{\partial}{\partial r}(r-r_1)^2e^{rt}$$

The order of differentiation can be interchanged by Clairaut's theorem.

$$\frac{\partial}{\partial r}L[e^{rt}] = a\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial r}(e^{rt}) + b\frac{\partial}{\partial t}\frac{\partial}{\partial r}(e^{rt}) + c\frac{\partial}{\partial r}e^{rt} = a\frac{\partial}{\partial r}(r-r_1)^2e^{rt}$$

Evaluate the derivatives.

$$\frac{\partial}{\partial r}L[e^{rt}] = a\frac{\partial^2}{\partial t^2}(te^{rt}) + b\frac{\partial}{\partial t}(te^{rt}) + cte^{rt} = a[2(r-r_1)e^{rt} + t(r-r_1)^2e^{rt}]$$

$$\frac{\partial}{\partial r}L[e^{rt}] = a(te^{rt})'' + b(te^{rt})' + cte^{rt} = 2a(r-r_1)e^{rt} + at(r-r_1)^2e^{rt}$$

Therefore,

$$\frac{\partial}{\partial r}L[e^{rt}] = L[te^{rt}] = 2a(r-r_1)e^{rt} + at(r-r_1)^2e^{rt}.$$

Setting $r = r_1$ in this equation results in

$$L[te^{r_1t}] = 0.$$

Setting $r = r_1$ in Eq. (i) results in

$$L[e^{r_1t}] = 0.$$

So both $y = e^{r_1t}$ and $y = te^{r_1t}$ are solutions to the ODE $L[y] = 0$.