Problem 23

In each of Problems 23 through 30, use the method of reduction of order to find a second solution of the given differential equation.

$$t^2y'' - 4ty' + 6y = 0, \quad t > 0; \qquad y_1(t) = t^2$$

Solution

Because this ODE is homogeneous, any constant multiple of $y_1(t)$ is also a solution: $cy_1(t) = ct^2$. According to the method of reduction of order, the general solution is obtained by allowing c to vary as a function of t.

$$y(t) = c(t)t^2$$

Substitute this formula for $y(t)$ into the ODE.

$$t^2[c(t)t^2]'' - 4t[c(t)t^2]' + 6[c(t)t^2] = 0$$

Evaluate the derivatives using the product rule.

$$t^2[c'(t)t^2 + 2c(t)t]' - 4t[c'(t)t^2 + 2c(t)t] + 6[c(t)t^2] = 0$$

$$t^2[c''(t)t^2 + 2c'(t)t + 2c'(t)t + 2c(t)] - 4t[c'(t)t^2 + 2c(t)t] + 6[c(t)t^2] = 0$$

$$t^4c''(t) + \cancel{4t^3c'(t)} + \cancel{2t^2c(t)} - \cancel{4t^3c'(t)} - \cancel{8t^2c(t)} + \cancel{6t^2c(t)} = 0$$

$$t^4c''(t) = 0$$

Divide both sides by t^4.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

$$c(t) = C_1t + C_2$$

Therefore, the general solution is

$$y(t) = (C_1t + C_2)t^2;$$

the second solution is $y_2(t) = t^3$.